Section II

5. (a) If V is a finite dimensional vector space over a field F, and V^* denotes the dual vector space of V, prove that $V \cong V^*$.

   (b) Let V be a finite dimensional vector space over a field F. For any subspace W of V let W^0 denote the annihilator of W. Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

6. (a) Let T be a linear operator on V, where V is as above. If M, N are matrices of T corresponding to two ordered bases of V, prove that the determinant of M equals the determinant of N.

   (b) Let V be the space of all $n \times 1$ column matrices over a field F. Show that every linear operator on V is left multiplication by a unique $n \times n$ matrix over F.

7. (a) Let T be a linear operator on a finite dimensional vector space V. Define characteristic and minimal polynomials of T and show that they have the same roots except for multiplicities.

   (b) Let T be a diagonalizable linear operator on $\mathbb{R}^5$ with set of characteristic values $\{1, 3, 7\}$. What will be the minimal polynomial of T? Justify your answer.

8. (a) For any linear operator T on a finite dimensional inner product space V with inner product $< \ , \ >$, show that there exists a unique linear operator T^* on V such that $< Tv, \overline{w} > = < v, T^*w >$ for all $v, w \in V$.

   (b) Let V be a finite dimensional complex inner product space. Let T be a linear operator on V. If T is unitary, prove that there is a basis of V with respect to which the matrix of T is diagonal.

Algebra - I

30 : 2ndH107

Con. 3798-07. KD-2205

External Scheme A] (3 Hours) [Total Marks : 100
External Scheme B] (2 Hours) [Total Marks : 40

N.B.(1) **Scheme B** students answer any **three** questions.
(2) **Scheme A** students answer any **five** questions.
(3) **All** questions carry **equal** marks.
(4) Write on the **top** of your answer-book the **scheme** under wich you are appearing.
(5) Answers to **both** the sections are to be written in the **same** answer-book.
(6) Throughout the paper, R denotes a commutative ring with identity and F denotes a field, unless otherwise stated. All rings considered are commutative rings with identity.

Section I

1. (a) Let G be a finite abelian group of order $p^e m$ where p is a prime that does not divide m. Show that G is the internal direct product of two subgroups H and K where $|H| = p^e$ and $|K| = m$.

   (b) Prove or disprove: Every abelian group of order 180 has a cyclic subgroup of order 18.

2. (a) Prove that the center of a group of order p^n, where p is a prime number and n is a natural number, is non trivial.

   (b) Let p be a prime number. Prove that a group of order p^2 is abelian.

3. (a) Show that $\mathbf{Z}[\omega]$, where ω is a primitive cube root of unity, is a Euclidean domain.

   (b) Let$R = \mathbf{Z}[\omega]$. Prove or disprove: $R[X]$ is a unique factorization domain.

4. (a) Let F be a field and let $p(X) \in F[X]$. Show that $< p(X) >$ is a maximal ideal in $F[X]$ if and only if $p(X)$ is irreducible over F.

   (b) Find all monic irreducible polynomials of degree 2 over $\mathbf{Z}/2\mathbf{Z}$.

[TURN OVER

Section II

5. (a) Let V be a vector space over a field F. If there exists an infinite subset of V which is linearly independent over F, prove that the diemnsion of V must be infinte.

   (b) If V is a finite dimensional vector space over a field F, and V^* denotes the dual vector space of V, prove that $V \cong V^*$.

6. (a) Let V be a vector space of dimension n over a field F. Fixing an ordered basis of V, prove that there is a one to one correspondence between linear operators on V and $n \times n$ matrices over F.

   (b) Let T be a linear operator on V, where V is as above. If M, N are matrices of T corresponding to two ordered bases of V, prove that determinant of M equals the determinant of N.

7. (a) Let V be a finite dimensional vector space over a field F and suppose characteristic of F is not equal to 2. If B is a non singular symmetric bilinear form on V, prove that there exists a basis of V with respect to which the matrix of B is diagonal.

   (b) Let $V = \mathbb{R}^2$. Let $B : V \times V \to \mathbb{R}$ be the symmetric bilinear form defined by $B((x_1, x_2), (y_1, y_2)) = x_1y_1 - x_2y_2$. Find the matrix of B with respect to the standard basis of V and determine its signature.

8. (a) Show that every complex $n \times n$ matrix is similar over $\mathbb{C}$ to an upper triangular matrix.

   (b) If A is a complex nilpotent matrix, prove that the eigen values of $I + A$ are all equal to 1.

M.A & M.Sc (Mathematics) (Part - I)

Mathematics : Paper - III - Topology

Con. 1692–09. **MS–8187**

For Internal (Scheme B)] (2 Hours) **[Total Marks : 40**

For External (Scheme A)] (3 Hours) **[Total Marks : 100**

20/4/09

N.B. : (1) Write on the **top** of your answer book the **Scheme** under which you are appearing.
(2) Students of **Scheme B** answer **three** questions with at least **one** question from **each section**; Students of **Scheme A** answer **five** questions with at least **two** questions from **each** section.
(3) **All** questions carry **equal** marks. Answers to **both** the sections are to be written in the **same** answer book.

Section I

1. (a) Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of finite sets. If J is finite, then show that the sets

$$\bigcup_{\alpha \in J} A_\alpha \quad \text{and} \quad \prod_{\alpha \in J} A_\alpha$$

are both finite sets.

(b) Show that for any non-empty set A, the cardinality of the power set of A is strictly greater than that of A.

2. (a) Define a basis and a subbasis for a topology on X. Give an example of a subbasis which is not a basis.

(b) Show that the countable collection $\mathbf{B} = \{ (a,b) : a, b \in Q \}$ is a basis for the standard topology on R, while the countable collection $\mathbf{B} = \{ (a,b) : a, b \in Q \}$ generates a topology different from the lower limit topology on R.

(c) Let X be a topological space. Let $\wedge$ denote the subset $\{ x \times x : x \in X \}$ of $X \times X$. Show that X is a Hausdorff space if and only if $\wedge$ is a closed subset of $X \times X$.

3. (a) State and prove the pasting lemma to prove continuity of a given function.

(b) Give an example of continuous bijection from one topological space to the other which is not a homeomorphism.

(c) Let $f : X \longrightarrow Y$ be a function where X is a metric space. Show that the function f is continuous iff for every convergent sequence $x_n \longrightarrow x$ in X the sequence $f(x_n)$ converges to f(x) in Y.

4. (a) Define a connected subspace of a topological space X. Show that if A is a connected subspace of X and if $A \subseteq B \subseteq \bar{A}$, then B is connected.

(b) Let $p \in X$ and let $\{ A_i : i \in I \}$ be a family of connected subsets of X such that $p \in A_i$ for every $i \in I$. Show that $\cup_{i \in I} A_i$ is connected.

(c) Show that product of two connected topological spaces is connected.

[TURN OVER

Section II

5. (a) Let Y be a subspace of X. Show that Y is compact iff every covering of Y by sets that are open in X has a finite subcollection covering Y.

   (b) Show that every closed subset of a compact space is compact.

   (c) Show that every locally compact Hausdorff space X which is not compact, has a one-point compactification Y such that Y is compact Hausdorff and $\bar{X}$ equals Y.

6. (a) Let X be the set R in the lower limit topology. Show that X is a Lindeloff space but $X \times X$ is not a Lindeloff space.

   (b) Show that a closed subspace of a normal space is normal.

7. (a) Let X be a metric space. Show that if every Cauchy sequence in X has a convergent subsequence, then X is complete.

   (b) Let X be a metric space. Show that X is compact iff X is a complete and totally bounded metric space.

   (c) Let C be the set of all continuous real valued functions on [0, 1] equipped with the sup metric. Let F be a subset of C. Show that if F is an equicontinuous family, then so is $\bar{F}$.

8. (a) Let $p : E \to B$ be a covering map. Let B be connected. Show that, if for some b_0, the set $p^{-1}(b_0)$ has k elements, then for every b, the set $p^{-1}(b)$ has k elements.

   (b) Let $p : E \to B$ be a covering map. Let $p(e_0) = b_0$. Show that any path $f : [0, 1] \to B$ beginning at b_0 has a unique lifting to a path $\bar{f}$ in E beginning at e_0.

Mathematics. Topology

Con. 4352–08. SM–4067

Scheme A] (3 Hours) [Total Marks : 100

N.B. : Answer any **four** questions.

Scheme B] (2 Hours) [Total Marks : 40

N.B. : Answer any three questions.

Topology

1. (a) Give an example with details to show that a countable product of countable sets need not be countable.

   (b) Let $\mathcal{A} = \{f \mid f : \mathbb{N} \longrightarrow \{0,1\}\}$ be the collection of all maps from $\mathbb{N}$ to $\{0,1\}$. Construct an injective map from $\mathbb{R}$ into $\mathcal{A}$.

2. (a) State and prove the 'Pasting Lemma'.

   (b) Let $f, g : [0,1] \longrightarrow X$ be continuous maps into a topological space X such that $f(1) = g(0)$. Define $h : [0,1] \longrightarrow X$ by $h(s) = f(2s)$ for all $s \in [0, \frac{1}{2}]$ and $h(s) = g(2s-1)$ for all $s \in [\frac{1}{2}, 1]$. Verify that h is a continuous map.

3. (a) Let X be a topological space. Prove that X is a connected space if and only if every continuous map from X to the discrete space $\{0,1\}$ is a constant function.

   (b) Define a path connected space. Prove that any open, connected subset of $\mathbb{R}^n$ is path connected.

4. (a) Prove that any compact subset of a Hausdorff space X is closed in X.

   (b) State and prove the 'Tube Lemma'.

5. Let X, Y, Z be topological spaces. S^1 denotes the unit circle in $\mathbb{R}^2$ with center at $(0,0)$

   (a) Let $f : X \longrightarrow Y$ be a map. When do you say f is a 'quotient map'. Show that $g : \mathbb{R} \longrightarrow S^1$ defined by $g(x) = (\cos x, \sin x)$ $(x \in \mathbb{R})$ is a quotient map.

   (b) Let $\eta : X \longrightarrow Y$ be a quotient map. Suppose $f : X \longrightarrow Z$, $g : Y \longrightarrow Z$ be maps with f being continuous and $g \circ \eta = f$. Then show that g is a continuous map.

6. (a) Define the 'interior of a subset' in a space. Define a 'Baire space'. Prove that a compact, Hausdorff space is a Baire space.

   (b) Prove that $\mathbb{Q}$ can not be written as intersection of countably many open subsets of $\mathbb{R}$.

7. (a) Define the notion of 'path homotopy'. Let $\alpha : [0,1] \longrightarrow X$ be a path in a space X with $\alpha(1) = q$. Prove that $\alpha * c_q$ is path homotopic to α $(c_q(s) = q$ for all $s \in [0,1])$.

   (b) Prove that $f : S^1 \longrightarrow S^1$ defined by $f(z) = z^2$ is a covering map.