M.A. & M.Sc. (PART - I)
Mathematics : paper III - Topology
21 st April 2008.

P4/RT-Ex-08-424

Con. 1335–08.

NB–5323

Scheme A] (3 Hours) **[Total Marks : 100**

N.B. : Answer any **four** questions.

Scheme B] (2 Hours) **[Total Marks : 40**

N.B. : Answer any **three** questions.

1. (a) Prove that a finite product of countable sets is countable.
   (b) Consider $\mathcal{A} = \{S \subset \mathbb{N} \mid S \text{ is an infinite set}\}$. Prove that $\mathcal{A}$ is an uncountable set.

2. (a) State and prove the 'Pasting Lemma'.
   (b) Consider the subsets A, B, C of $\mathbb{R}^2$ defined by
   $A = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\}$
   $B = \{(x, y) \in \mathbb{R}^2 \mid (x-1/2)^2 + y^2 = 1/4\}$
   and $C = \{(x, y) \in \mathbb{R}^2 \mid (x+1/2)^2 + y^2 = 1/4\}$.
   Construct a continuous bijection from $A \cup B$ onto $A \cup C$.

3. (a) Give an example of a connected space which is not path connected. Justify.
   (b) Prove that $\mathbb{R}^n \setminus \{0\}$ $(n > 1)$ is connected.

4. (a) State and prove the 'Tube Lemma'.
   (b) Let X, Y be topological spaces. If Y is compact, then prove that the projection $\pi_1 : X \times Y \longrightarrow X$ is a closed map.

5. (a) Define the terms: Second Countable Space, Separable Space. Prove that a second countable space is separable.
   (b) Find a countable dense subset of the irrational numbers. Justify.

6. (a) Prove that there does not exist a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is continuous only at rational numbers.
   (b) Prove that a locally compact, Hausdorff space is a Baire space.

7. (a) Let $p : E \longrightarrow B$ be a covering map. Assume B is path connected. For any $x, y \in B$, prove that $p^{-1}(x), p^{-1}(y)$ have same cardinality.
   (b) Define $c(t) = (\cos 2\pi t, \sin 2\pi t) \in S^1$ $(t \in [0, 1])$. Show that c is not path homotopic to a constant path in the space S^1.

M.Sc (Mathematics) - I - ~~Analysis~~ Topology / ①

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rnal (Scheme B)	(2 Hours)	[Total Marks : 40
ternal (Scheme A)	(3 Hours)	[Total Marks : 100

N.B. (1) Scheme-B students answer any three questions selecting atleast one from each section.
(2) Scheme-A students answer any five questions selecting atleast two from each section.
(3) All questions carry equal marks.
(4) Write on the top of your answer book the scheme under which you are appearing.
(5) Answers to both the sections are to be written in the same answer book.

SECTION I

1(a) Define a countable set. A and B are countable sets. Prove that $A \times B$ is also countable.

1(b) Lat S be the set of sequences $s = (s_n)$ such that $s_n \in \{0, 1\}$ for all $n \in \mathbb{N}$. Show that S is not countable.

2(a) Let X be a topological space. Let $A \subset X$. Define ∂A-the boundry of A. Prove that ∂A is empty if and only if A is both open and closed.

2(b) Show that a separable metric space is second countable.

3(a) Let $f : X \to Y$ be a map of topological spaces. Prove that the fllowing two conditions are the equivalent: (i) $f^{-1}(F)$ is a closed set for every closed subset F of Y. (ii) $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.

3(b) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (x, |y|)$. Prove that f is a closed map.

4(a) Define a ' quotient map'. Suppose $\eta : X \to Y$ is a quotient map. Suppose $f : X \to Z$ and $g : Y \to Z$ are maps of topological spaces such that $g \circ \eta = f$. Show that f is continuous if and only if g is continuous.

4(b) If (X_1, τ_1) and (X_2, τ_2) are topological spaces, define product topology on $X = X_1 \times X_2$. Prove that the produc space X is separable if and only if both X_1 and X_2 are separable.

SECTION II

5(a) Prove that $[0, 1]$ is a compact subset of $\mathbb{R}$ provided with ususal topology.

5(b) Prove thayt every continuous map $f : X \to \mathbb{R}$ on a compact metric space X is uniformly continuous, bounded and attains the bounds.

6(a) Prove that every open subset of $\mathbb{R}$ can be written as a countable disjoint union of open intervals.

6(b) Find the connected components of $\{(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\} \mid xy = 0\}$

7(a) Let α, β and γ be loops in the topological space X based at a point $a \in X$. Define $\alpha * \beta$. Prove that $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$ are path homotopic.

7(b) Let $p : G \to B$ be a covering map. Assume that B is path connected. Show that there is a bijection from $p^{-1}(a)$ and $p^{-1}(b)$ for any two points a and b in B.

8(a) Consider the covering map $p : S^1 \to S^1$ defined by $p(z) = z^2$. Define $\gamma : [0, 1] \to S^1$ by $\gamma(s) = \cos(\pi s) + \sqrt{-1}\sin(\pi s)$. Find a path $\mu : [0, 1] \to S^1$ such that $\mu(0) = -1$ and $(p \circ \mu)(s) = \gamma(s)$ for all $s \in [0, 1]$.

8(b) Let X be a topological space and $A \subseteq X$. When do you say A is a retract of X? Prove that S^1 is a retract of $\mathbb{R}^2 \setminus \{(0, 0)\}$.

M.A. / M.Sc. (Mathematics) (Part - I

V Ex-I-09-C-19

Mathematics Paper - IV - Complex Analysis

Con. 1433-09. **MS-6987**

27/4/09

Scheme A] (3 Hours) **[Total Marks : 100**

Scheme B] (2 Hours) **[Total Marks : 40**

N.B. : (1) **All** questions carry **equal** marks.

(2) Candidates of **Scheme A** should attempt any **five** questions.

(3) Candidates of **Scheme B** should attempt any **three** questions.

1. (a) Prove that all points $z \in \mathbb{C}$ satisfying—

$$\left|\frac{z+1}{z+4}\right| = 2$$

lie on a circle. Find its centre and radius.

(b) let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$. Prove that their cross ratio is real if and only if the four points lie on a circle.

2. (a) Let w be an n-th root of unity. If $w \neq 1$, show that—

$$1 + w + w^2 + \ldots + w^{n-1} = 0.$$

(b) Find all the fourth roots of $8 + i8\sqrt{3}$.

(c) Define the principal branch of the complex logarithm. Evaluate i^i, taking the logarihm in its principal branch.

3. (a) Prove that the real and imaginary parts of a holomorphic function satisfy the Cauchy-Riemann equations in its domain of holomorphy.

(b) For the function f(z) defined by—

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0. \end{cases}$$

prove that the Cauchy-Riemann equations are satisfied at the origin, but the function f(z) is not differentiable at the origin.

4. (a) Show that the power series $\sum_{j=0}^{\infty} a_j z^j$ and $\sum_{j=k}^{\infty} j(j-1)\ldots(j-k+1)\ a_j z^{j-k}$ have the same radius of convergence for $k \in \mathbb{N}$.

(b) Find the radius of convergence of—

(i) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} (z/2)^{2n}$ (ii) $\sum_{j=2}^{\infty} z^{2j} / j(j-1)$

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5. (a) State and prove the Cauchy theorem for a triangle.
   (b) Using the Cauchy integral formula evaluate—

$$\int_{\Gamma} \frac{\sin \pi z}{z^2 + 1}$$

   where Γ is the circle $|z| = 3$.

6. (a) State and prove the open mapping theorem for a holomorphic function.
   (b) Show that a holomorphic function which only takes real values is a constant.

7. (a) Prove that a complex polynomial of degree n has exactly n zeros.
   (b) Prove that the zeros of a holomorphic function are isolated.

8. (a) State and prove Rouché's theorem.
   (b) Determine the number of zeros, counting multiplicities, of the polynomial $z^4 - 2z^3 + 9z^2 + z - 1$ inside the circle $|z| = 2$.

9. (a) Let f(z) be an analytic function with an essential singularity at $z = a$. For any given complex number w, show that there exists a sequence $\{ z_n \}$ converging to a such that the sequence $\{ f(z_n) \}$ converges to w.
   (b) Use the Cauchy Residue Theorem to evaluate the real integral—

$$\int_0^{\infty} \frac{\cos x}{x^2 + 1}.$$

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Con. 4356–08. M.A & M.Sc (Part-I) SM–4062

Mathematics. P. IV — Complex Analysis

Scheme A (External) (3 Hours) Total Marks : 100

20/10/08

Scheme B (Internal) (2 Hours) [Total Marks : 40

N.B: **Scheme A (External)** students should attempt **five** questions selecting at least **two** from **each section.**

Scheme B (Internal) students should attempt **three** questions selecting at least **one** from **each section.**

Answers to both the sections are to be written in the same answerbook.

All questions carry equal marks.

Note: $\mathbb{C}$ denotes the set of all complex numbers. A holomorphic function on an open set $G \subset \mathbb{C}$ is a complex function differentiable at every point of G.

Section I

Q. 1 a] Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series and let r be given by

$$\frac{1}{r} = \limsup |a_n|^{\frac{1}{n}} \quad , \quad 0 < r \leq \infty .$$

Prove : (i) if $|z| < r$, the series converges absolutely;

(ii) if $0 < \rho < r$ then the series converges uniformly on $\{z : |z| \leq \rho\}$.

b] Find the radius of convergence of the series

(i) $\sum_{n=0}^{\infty} n^n z^n$ (ii) $\sum_{n=0}^{\infty} \frac{z^n}{n}$

Q. 2 a] Let $G \subset \mathbb{C}$ be open and let $f : G \to \mathbb{C}$ be defined by $f(z) = u(z) + iv(z)$ where u and v are real valued functions defined on G. If u and v have continuous partial derivatives and if they satisfy the Cauchy-Riemann equations then prove that f is complex differentiable at every point of G.

b] Find a holomorphic function $f(z) = u(z) + iv(z)$ whose real part is $2xy + 2x$.

Q. 3 a] Prove that a Mobius transformation takes circles onto circles.

b] Find the image of the circle $x^2 + y^2 + 2x = 0$ in the complex plane under the transformation $w = \frac{1}{z}$.

Q. 4 a] If $G \subset \mathbb{C}$ is open and connected and f is a branch of $\log z$ on G, prove that the totality of branches of $\log z$ are the functions $f(z) + 2\pi ni$, $n \in Z$ (set of integers).

b] Prove that the function $f(z)$ is not complex differentiable at any point z in the complex plane $\mathbb{C}$ if $f(z) = \bar{z} = x - iy$.

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