Section II

Q. 5 a] State and prove Cauchy's theorem for a triangle.

b] Use Cauchy-integral formula to evaluate

$$\int_{\gamma} \frac{\cos(e^z)}{z(z+2)}\,dz \qquad \text{where } \gamma \text{ is the circle } |z| = 1.$$

Q.6a] Let $D = \{z : |z| < 1\}$ be the unit disk and suppose f is holomrphic on D with $f(0) = 0$ and $|f(z)| < 1$ for z in D. Prove that $|f'(0)| \le 1$ and $|f(z)| \le |z|$ for all z in D.

b] Let α be a complex number in the unit disk $D = \{z : |z| < 1\}$. Find a one-one holomorphic function from D onto itself taking α to 0.

Q. 7 a] State and prove Casorati-Weierstrass theorem.

b] Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for

(i) $1 < |z| < 2$, (ii) $|z| < 1$.

Q. 8 a] State and prove Residue theorem.

b] Use Residue theorem to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+4}$$

****

VT March 08 234

Complex Analysis

Con. 1045–08. NB–4240

Scheme A (External) (3 Hours) [Total Marks : 100

Scheme B (Internal) (2 Hours) [Total Marks : 40

N.B: **Scheme A (External)** students should attempt **five** questions selecting at least **two** from **each section.**
Scheme B (Internal) students should attempt **three** questions selecting at least **one** from **each section.**
Answers to both the sections are to be written in the same answerbook.
All questions carry equal marks.

Note: $\mathbb{C}$ denotes the set of all complex numbers. A holomorphic function on an open set $G \subset \mathbb{C}$ is a complex function differentiable at every point of G.

Section I

Q. 1 a] If the power series $\sum a_n z^n$ converges for a particular value $z_0 (\neq 0)$ of z, prove that it converges (absolutely) for every z for which $|z| < |z_0|$.

b] Prove that the power series $\sum_{n=0}^{\infty} \frac{(z+2)^n}{(n+2)^3 4^{n+1}}$ converges for every z such that $|z+2| < 4$.

Q. 2 a] Let $f(z) = u(z) + iv(z)$ be a holomorphic function on a domain G, prove that $f(z)$ satisfies the Cauchy-Riemann equations on G.

b] Suppose $f : G \to \mathbb{C}$ is holomorphic and that G is connected. Show that if $f(z)$ is real for all z in G then f is constant.

Q. 3 a] Define a Mobius transformation. If z_2, z_3, z_4 are distinct points in $\mathbb{C}_\infty$ and T is any Mobius transformation, prove that

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4) \quad \text{for any point } z_1.$$

b] Find a Mobius transformation which maps the points $z = 1, i, -1$ onto the points $w = 0, 1, \infty$. Also find the image of the circle $|z| = 1$ in the complex plane under this Mobius transformation.

Q. 4 a] Find the fourth roots of $(-1+i)$ and locate them graphically.

b] Prove that $|\sin z|^2 = \sin^2 x + \sinh^2 y$.

c] Define the function $\cos z$ and prove that it is entire function.

[TURN OVER

Section II

Q. 5a] Let Ω be a star shaped domain with respect to α and f be an holomorphic function on Ω. Prove that there exists a holomorphic function F on Ω such that $F'(z) = f(z)$ in Ω.

b] Use Cauchy-Integral formula to evaluate

$$\int_{\gamma} \frac{\cos \pi z}{z^2 - 1}\, dz \qquad \text{where } \gamma \text{ is the circle } |z-1| = 1.$$

Q. 6a] If f is bounded entire function, prove that f is constant.

b] Let $f(z) = u(x,y) + i\,v(x,y)$ where $z = x + iy$ be an entire function. If $u(x,y) = Re\, f(z)$ is bounded for all z in complex plane $\mathbb{C}$, prove that $u(x,y)$ and $v(x,y)$ are constant functions.

Q. 7 a] State and prove Rouche's theorem.

b] Determine the number of zeros, counting multiplicities, of the polynomial $z^4 + 3z^3 + 6$ inside the circle $|z| = 2$.

Q. 8 a] Define: Isolated singularity. Let $z = z_0$ be an isolated singularity of f and let

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$$ be its Laurent series in $ann(z_0; 0, R)$. Prove that $z = z_0$ is a removable singularity of f iff $a_n = 0$ for $n \le -1$.

b] Use Cauchy-Residue theorem to evaluate

$$\int_0^{\infty} \frac{dx}{x^2 + 1}$$

****

Complex Analysis

P3/D/UPQ/-OLE07-9-12 47

Con. 3842-07. **KD-2238**

Scheme A (External) (3 Hours) [Total Marks : 100

Scheme B (Internal) (2 Hours). [Total Marks : 40

N.B: Scheme A(External) students should attempt **five** questions selecting at least **two** from **each section**.
Scheme B(Internal) students should attempt **three** questions selecting at least **one** from **each section**.
Answers to both the sections are to be written in the same answerbook.
All questions carry equal marks.

Note: A holomorphic function on an open set $G \subset \mathbb{C}$ is a complex function differentiable at every point of G.

Section I

1. (a) Define: The cross ratio (z_1, z_2, z_3, z_4) of z_1, z_2, z_3 and z_4. If z_2, z_3, z_4 are distinct points in $\mathbb{C}_\infty$ and T is any Mobius transformation, prove that for any point z_1

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4).$$

   (b) Let $H = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ and let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Find a Mobius transformation g such that $g(H) = D$ and $g(i) = 0$. Justify your claims.

2. (a) If $\sum_{n=0}^{\infty} a_n(z-a)^n$ is a given power series with radius of convergence R, then prove that

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

   if this limit exists.

   (b) Show that the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ is 1. Discuss the convergence of this series at points on the boundary of the disc $\{z \in \mathbb{C} \mid |z| \leq 1\}$.

3. (a) Construct a branch of logarithm $l(z)$ on $G = \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$. If f is any other branch of logarithm on G then show that there exists an integer k such that $f(z) = l(z) + 2\pi ik$ for all z in G.

   (b) Compute the following values when $\log z$ is defined by its principal value on the open set $G = \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$.
   (i)i^i (ii)2^i.

4. (a) Let G be either the whole complex plane $\mathbb{C}$ or some open disc in $\mathbb{C}$. If $u : G \to \mathbb{R}$ is a harmonic function, prove that u has a harmonic conjugate.

   (b) If G is an open subset of $\mathbb{C}$ and if $u : G \longrightarrow \mathbb{R}$ is a harmonic function, prove that u is infinitely differentiable.

   (c) Let a function f be holomorphic on an open connected subset G of $\mathbb{C}$. If $f(z)$ is real for all z in G then prove that f must be constant on G.

[TURN OVER

Section II

5. (a) State and prove Cauchy's Theorem for a triangle.

   (b) Use Cauchy's integral formula to evaluate

$$\int_{\gamma} \frac{\sin z}{(z-\pi)(z-(\pi/2))}\, dz \qquad \text{where } \gamma \text{ is the circle } |z| = 2.$$

6. (a) Let f be a holomorphic function on an open connected subset G of $\mathbb{C}$. If there is a point $a \in G$ such that $f^{(n)}(a) = 0$ for $n = 0, 1, 2, \cdots$ then prove that f is identically zero on G.

   (b) Let f be an entire function. Suppose there is a constant M, an $R > 0$ and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree $\leq n$.

7. (a) Define : (i) Removable singularity (ii) Essential singularity

   (b) Let A_δ be the annulus $\{z \in \mathbb{C} \mid 0 < |z-a| < \delta\}$ and let f be a function holomorphic on A_δ. If f has an essential singularity at a then show that $f(A_\delta)$ is dense in $\mathbb{C}$.

   (c) Expand

$$f(z) = \frac{e^{2z}}{(z-1)^3}$$

   in a Laurent series about $z = 1$ and name the singularity.

8. (a) State Riemann Mapping Theorem.

   (b) State and prove Argument Principle.

   (c) Use Residue Theorem to evaluate

$$\int_0^{\infty} \frac{dx}{(x^2+1)^2}.$$

External (Scheme A)] (3 Hours) [Total Marks : 100

Internal/External (Scheme B)] (2 Hours) [Total Marks : 40

4/5/09

N.B. (1) **Scheme-A** students answer any **five** questions.

(2) **Scheme-B** students answer any **three** questions.

(3) **All** questions carry **equal** marks.

(4) **Write** on the **top** of your **answer book** the **scheme** under which you are appearing.

1. (a) Find the number of 3-element subsets {a, b, c} of {1, 2, ... , 2008} such that 3 divides a + b + c.

(b) 6 boys and 5 girls are to be seated around a table. Find the number of ways this can be done such that no two girls are adjacent.

2. (a) Show that $\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$.

(b) How many minimum numbers must be selected from the set {1, 2, 3, 4, 5, 6} to guarantee that at least one pair of these numbers add to 7 ?

3. (a) For each $n \in \mathbb{N}$, show that the number of partitions of n into parts each of which appears at most twice, is equal to the number of partitions of n into parts the sizes of which are not divisible by 3.

(b) For each $n \in \mathbb{N}$, let a_r denote the number of r-digit ternary sequence that contain an odd number of 0's and an even number of 1's. Find a_r.

4. (a) Define matching in the bipartite graph $G = (X \cup Y, E)$. Show that the bipartite graph $G = (X \cup Y, E)$ has a complete matching if and only if $|J(A)| \geq |A|$ for all $A \subseteq X$, where $J(A) = \{ y \in Y / xy \in E \text{ for some } x \in A \}$.

(b) Find the largest number of sets in the family $A_1, A_2, \ldots, A_{10}$ which together have a system of distinct representatives, where $A_1 = \{1, 8, 10, 13\}$, $A_2 = \{1, 4, 5, 7, 11\}$, $A_3 = \{5, 8\}$, $A_4 = \{8, 13\}$, $A_5 = \{2, 3, 4, 11, 12\}$, $A_6 = \{5, 6, 10, 13\}$, $A_7 = \{10, 13\}$, $A_8 = \{5, 8, 10, 13\}$, $A_9 = \{1, 5, 8\}$, $A_{10} = \{1, 5, 8, 10, 13\}$.

5. (a) Define D_n, derangement of n objects. Show that —

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$$

(b) Solve recurrence relation $a_n - 3a_{n-1} = 2 - 2n^2$ with initial condition $a_0 = 3$.

6. (a) Let G be a group of permutations of a non empty finite set X, and let x be any chosen element of X. Then show that $|Gx| \times |G_x| = |G|$ where $Gx = \{g(x) / g \in G\}$ and $G_x = \{g \in G / g(x) = x\}$.

(b) What is the smallest number of colors required to color faces cubes to get 57 different colored cubes ?

7. (a) Calculate the number of words that can be formed by rearranging the letters ALLAHABAD so that no letters appears at one of its original positions.

(b) State and prove Mobius inversion formula.

8. (a) You have three coins in your pocket, two fair ones but the third biased with probability of heads p and tails 1 – p. One coin selected at random drops to the floor, landing heads up. How likely is it that it is one of the fair coins ?

(b) For independent random variables X and Y, show that E(XY) = E(X)E(Y), where E(XY), E(X) and E(Y) denote expectations of XY, X and Y respectively.

Mha I -07 438

Combinatorics

NB-6472

C[redacted]669-08.

External (Scheme A)] (3 Hours) [Total Marks: 100

[redacted]ernal/External (Scheme B)] (2 Hours) [Total Marks: 40

N.B. 1) Scheme-A students answer any five questions.
2) Scheme-B students answer any three questions.
3) All questions carry equal marks.
4) Write on the top of your answer book the scheme under which you are appearing.

1. (a) How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other?
   (b) What is the probability that each player has a hand containing an ace when the 52 cards of a standard deck are dealt to four players?

2. (a) During a month with 30 days a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 45 games.
   (b) Prove that for integer m, $0 \leq m \leq \binom{n}{r} - 1$, can be uniquely expressed as

$$m = \sum_{i=1}^{r} \binom{v_i - 1}{i}, \text{ for some integers } 1 \leq v_1 \leq v_2 \leq \ldots \leq v_r \leq n.$$

3. (a) How many different strings can be made from the letters in MISSISSIPPI, using all letters?
   (b) Determine the number of different ways ten identical balloons can be given to four children if each child receives at least two balloons.

4. (a) Define Stirling numbers of second kind $S(n,k)$ and prove the following identity:

$$x^n = \sum_{k=1}^{n} S(n,k)[x]_k, \text{ where } [x]_k \text{ denotes the falling factorial.}$$

   (b) Let $R_{n,m}$ be the rook polynomial of $n \times m$ chess board. Derive a recurrence relation connecting $R_{n,m}, R_{n-1,m}, R_{n-1,m-1}$.

5. (a) Solve the recurrence relation $a_k = 4a_{k-1} - 4a_{k-2} + k^2$ with initial conditions $a_0 = 2$ and $a_1 = 5$.
   (b) State and prove Mobius inversion formula.

6. (a) Let $G = (X \cup Y, E)$ be a bipartite graph. Show that the largest number of edges in a matching in G equals the smallest number of vertices that cover the edges.
   (b) G be a group of permutations of a set X. Show that number of orbits of G on X is $\frac{1}{|G|}\sum_{g \in G} |F(g)|$ where $F(g) = \{x \in X / g(x) = x\}$.

7. (a) Find the cycle index of a group of symmetries of a regular pentagon. Hence find the number of ways of colouring exactly two vertices black, one white and two red.
   (b) Show that the number of partitions of a positive integer n into at most k parts is equal to the number of partions of n in which every part is less than or equal to k.

8. (a) Let X and Y be independent random variables with binomial distribution $B(m,p)$ and $B(n,p)$, respectively. What is the distribution of $X + Y$.
   (b) You have two coins, a fair one with probability of heads $\frac{1}{2}$ and an unfair one with probability of heads $\frac{1}{3}$, but otherwise identical. A coin is selected at random ar tossed, falling heads up. How likely is its that it is the fair coin.

M.A. MSC (Part - II) (Paper - V) Combinatorics

Mathematics.

Con. 4360–08. **SM–4055**

External (Scheme A)] (3 Hours) [Total Marks: 100
Internal/External (Scheme B)] (2 Hours) [Total Marks: 40

N.B. 1) Scheme-A students answer any five questions.
2) Scheme-B students answer any three questions.
3) All questions carry equal marks.
4) Write on the top of your answer book the scheme under which you are appearing.

1. (a) How many positive integers between 100 and 999 both inclusive are not divisible by either 3 or 4?
(b) Let m, n and r be non-negative integers with r not exceeding either m or n. Show that $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$.

2. (a) Show that every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.
(b) How many solutions are there to the inequality $x_1+x_2+x_3 \leq 11$ where x_1, x_2 and x_3 are non-negative integers ?

3. (a) How many 10 letters words are there in which each of the letter e and n occur at least once and each of the letter r, s occur at least twice.
(b) Define Stirling number $S(n,k)$ of second kind for $1 \leq k \leq n$. Show that $S(n,1) = 1 = S(n,n)$ and $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ for $2 \leq k \leq n-1$.

4. (a) Let X be finite set with n number of elements. Show that number of subset of X with even number of elements is same as number of subsets of X with odd number of elements.
(b) Find number of primes less than 200 using the principal of inclusion exclusion.

5. (a) In how many ways can 25 identical donuts be distributed to four police officers so that each officer gets at least three but no more than seven donuts?
(b) In how many ways can a $2 \times n$ rectangular board be tiled using 1×2 and 2×2 pieces?

6. (a) Define matching M in the bipartite graph $G = (X \cup Y, E)$. When M is said to be a maximum matching? Show that if M is not a maximum matching then G contains alternating path for M.
(b) Show that $r \times n$ Latin rectangle can always be completed to a Latin square of order n.

7. (a) How many round necklaces can be constructed using two red beads, three white beads and two blue beads?
(b) In how many ways can Rs. 100 be exchanged for notes of the value Rs. 50, Rs. 20, Rs. 10 and Rs. 5?

8. (a) A student has to sit for an examination consisting of 3 questions selected randomly from a list of 100 questions. To pass, he needs to answer all three questions. What is the probability that the student will pass the examination if he knows the answers to 90 questions on the list?
(b) State and prove Bayes' Theorem.

Con. 3758–07.

KD–2223

External (Scheme A)] (3 Hours) [Total Marks : 100

Internal / External (Scheme B)] (2 Hours) [Total Marks : 40

N.B. (1) Scheme-A students answer any **five** questions.
(2) Scheme-B students answer any **three** questions.
(3) All questions carry equal marks.
(4) Write on the top of your answer book the scheme under which you are appearing.

1. (a) Find the number of ways of choosing 3 distinct integers frm the set $X = \{1, 2, \ldots, 100\}$ so that their sum will be divisible by 3.
   (b) Determine the total number of combinations (of any size) of a multiset of k distinct objects with finite repetition numbers $n_1, n_2, \ldots, n_k$ respectively.

2. (a) Consider the set of words of length n generated from the alphabet $\{0, 1, 2\}$.
   i. Show that the number of words in each of which the digit 0 appears an even number of times is $\frac{3^n+1}{2}$.
   ii. Using (i) above or otherwise, prove the identity:
   $$\binom{n}{0}2^n + \binom{n}{2}2^{n-2} + \cdots + \binom{n}{q}2^{n-q} = \frac{3^n+1}{2},$$
   where $q = n$ if n is even and $q = n-1$ if n is odd.
   (b) Use a combinatorial argument to prove that $\frac{(2n)!}{2^n}$ and $\frac{(3n)!}{3^n \times 2^n}$ are integers.

3. (a) Prove that of any 10 points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $\frac{1}{3}$.
   (b) Find the number of positive integral solutions of
   $$(x_1 + x_2 + x_3)(y_1 + y_2 + y_3 + y_4) = 77.$$

4. (a) Find the number of permutations of the nine positive digits in which the blocks 415, 12, and 23 do not appear.
   (b) Given a sequence of $2n$ elements, find the number of their derangements such that the first n elements of each derangement are (i) the first n elements of the sequence, and (ii) the last n elements of the sequence.

5. Attempt any TWO.
   (a) Using an argument based on Ferrer diagrams or otherwise show that the number of partitions of n which have at most m parts is equal to the number of partitions of $n + \frac{1}{2}m(m+1)$ in which there are m parts, all of them different.
   (b) Explain what is meant by Hypergeometric random variable and calculate its expectation and variance.
   (c) Solve the recurrence relation: $a_n = \sum_{k=1}^{n-1} a_k a_{n-k}$, $n \geq 2$ subject to initial value $a_1 = 1$.

6. (a) For two independent throws of a balanced die, find the expectations of the following variables:
   i. Twice the larger score minus the second.
   ii. The total number of fours and sixes.
   (b) In a sample 2% of the population have a certain blood disease in a serious form; 10% have it in a mild form; and 88% don't have it at all. A new blood test is developed; the probability of testing positive is $\frac{9}{10}$ if the subject has the serious form, $\frac{6}{10}$ if the subject has the mild form, and $\frac{1}{10}$ if the subject doesn't have the disease. I have just tested positive. What is the probability that I have the serious form of disease?

[TURN OVER

7\. (a) Let $A_1, \cdots, A_n$ be n subsets of a set S such that $|A_i| = m$ for each $i, 1 \leq i \leq n$ and such that each element of S occurs in exactly m of the sets from $A_1, \ldots, A_n$. Use Hall's theorem to show that the sets $A_1, \ldots, A_n$ posses a system of distinct representatives.

(b) Define $S(n, k)$,the Stirling numbers of second kind. Calculate : (i)$S(n, 2)$, (ii)$S(n, n-1)$ and (iii) $S(7, 3)$.

8\. (a) State and prove Burnside Frobenius theorem about the number of orbits under an action of a finite group acting on a finite set.

(b) Find the number of distinguishable necklaces made of of 9 spherical stones using three colors. How many of these use 2 blue, 3 green and 4 red stones?
