Order and Chaos in Nature
Course notes

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Contents

Chapter 1

A bit of history of chaos

1.1 Newton [1642-1727]

Newton's Laws: The equation of motion for a particle of mass m under a force field $\mathbf{F}(\mathbf{x}, t)$ is given by

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}(\mathbf{x}, t)$$

Given initial condition $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$, we can determine $\mathbf{x}(t)$ *in principle.*

Using Newton's laws we can understand dynamics of many complex dynamical systems, and predict their future quantitively. For example, the equation of a simple oscillator is

$$m\ddot{x} = -kx$$

whose solution is

$$x(t) = A\cos(\sqrt{k/m}t) + B\sin(\sqrt{k/m}t),$$

with A and B to be determined using initial condition. The solution is simple oscillation.

Planetary motion (2 body problem)

$$\mu\ddot{\mathbf{r}} = -(\alpha/r^2)\hat{\mathbf{r}},$$

the solution is elliptical orbit for the planets. In fact the astronomical data matched quite well with the predictions. Newton's laws could explain dynamics of large number of systems, e.g., motion of moon, tides, motion of planets, etc.

1.2 Laplace [1749-1827]- Determinism

Newton's law was so successful that the scientists thought that the world is deterministic. In words of Laplace

"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at any given moment knew all of the forces that animate nature and the mutual positions of the beings that compose it, if this intellect were vast enough to submit the data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom; for such an intellect nothing could be uncertain and the future just like the past would be present before its eyes."

1.3 Poincare [1854-1912]-Chaos in Three-Body Problem

One of the first glitch to the dynamics came from three-body problem. The question posed was whether the planetary motion is stable or not. It was first tackled by Poincare towards the end of ninteenth century. He showed that we cannot write the trajectory of a particle using simple function. In fact, the motion of a planet could become random or disorderly (unlike ellipse). This motion was called chaotic motion later. In Poincare's words itself.

"If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. but even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon. - in a 1903 essay "Science and Method"."

Clearly determinism does not hold in nature in the classical sense.

1.4 Fluid Motion- Weather Prediction [1950]

Motion of fluid parcel is given by

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \nu \nabla^2 \mathbf{u}.$$

where ρ, $\mathbf{u}$, and p are the density, velocity, and pressure of the fluid, and ν is the kinetic viscosity of the fluid. The above equation is Newton's equation for fluids. There are some more equations for the pressure and density. These complex set of equations are typically solved using computers. The first computer solution was attempted by a group consisting of great mathematician named Von Neumann. Von Neumann thought that using computer program we could predict weather of next year, and possibly plan out vacation accordingly. However his hope was quickly dashed by Lorenz in 1963.

1.5 Lorenz - Reincarnation of Chaos

In 1961, Edward Lorentz discovered the butterfly effect while trying to forecast the weather. He was essentially solving the convection equation. After one run, he started another run whose initial condition was a truncated one. When he looked over the printout, he found an entirely new set of results. The results was expected to be same as before.

Lorenz believed his result, and argued that the system is sensitive to the initial condition. This accidental discovery generated a new wave in science after a while. Note that the equations used by Lorenz do not conserve energy unlike three-body problem. These two kinds of systems are called dissipative and conservative systems, and both of them show chaos.

1.6 Robert May - Chaos in Population Dynamics

In 1976, May was studying population dynamics using simple equation

$$P_{n+1} \quad = \quad aP_n(1 - P_n)$$

where P_n is the population on the nth year. May observed that the time series of P_n shows constant, periodic, and chaotic solution.

1.7 Universality of chaos and later developments

In 1979, Feigenbaum showed that the behaviour of May's model for population dynamics is shared by a class of systems. Later scientists discovered that these features are also seen in many experiments. After this discovery, scientists started taking chaos very seriously. Some of the pioneering experiments were done by Gollub, Libchaber, Swinney, and Moon.

1.8 Deterministic Chaos- Main ingradients

- Nonlinearity: Response not proportional to input forcing (somewhat more rigourous definition a bit later
- Sensitivity to initial conditions.
- Deterministic systems too show randomness (deterministic chaos). Even though noisy systems too show many interesting stochastic or chaotic behaviour, we will focus on deterministic chaos in these notes.

1.9 Current Problems of Interest

- A few degrees of system (3 to 6) to Many degrees of systems
- Chaotic dynamics: Temporal variation of total population, but systems with many degrees of freedom show complex behaviour including spatiotemporal phenomena.. Complex Spatiotemporal behaviour is seen in convection and fluid flows. Turbulent behaviour is observed for even higher forcing.

1.10 A word on Quantum Mechanics

In QM the system is described by wavefunction. The evolution of the wavefunction is deterministic. However in QM one cannot measure both position and velocity precisely. There is uncertainty involved all the time

$$\Delta x \Delta p \geq h.$$

Hence, even in QM the world is not deterministic as envisaged by classical physicists. In the present course we will not discuss QM.

Quantum chaos is study of classical systems that shows chaos.

1.11 Pictures (source: wikipedia)

Newton, Laplace, Poincare, Lorenz

1.12 References

- H. Strogatz, Nonlinear Dynamics and Chaos, Levant Books (in India).
- R. C. Hilborn, Chaos and Nonlinear Dynamics, Oxford Univ. Press.

Chapter 2

Dynamical System

2.1 Dynamical System

A dynamical system is specified by a set of variables called state variables and evolution rules. The state variables and the time in the evolution rules could be discrete or continuous. Also the evolution rules could be either deterministic or stochastic. Given initial condition, the system evolves as

$$\mathbf{x}(0) \rightarrow \mathbf{x}(t).$$

The objectives of the dynamical systems studies are to devise ways to characterize the evolution. We illustrate various types of dynamical systems in later sections using some examples.

The evolution rules for dyanamical systems are quite precise. Contrast this with psychological laws where the rules are not precise. In the present course we will focus on dynamical systems whose evolution is deterministic.

2.2 Continuous state variables and continuous time

The most generic way to characterize such systems is through differential equations. Some of the examples are

1. **One dimensional Simple Oscillator**: The evolution is given by

   $$m\ddot{x} = -kx,$$

   We can reduce the above equation to two first-order ODE. The ODEs are

   $$\begin{aligned} \dot{x} &= p/m, \\ \dot{p} &= -kx. \end{aligned}$$

   The state variables are x and p.

2. **LRC Circuit:** The equation for a LRC circuit in series is given by

   $$L\frac{dI}{dt} + RI + \frac{Q}{C} = V_{applied}.$$

The above equation can be reduced to

$$\begin{aligned}\dot{Q} &= I, \\ L\dot{I} &= V_{applied} - RI - \frac{Q}{C}.\end{aligned}$$

The state variables are Q and I.

3. **Population Dynamics:** One of the simplest model for the volution of populution P over time is given by

$$\dot{P} = \alpha P - P^2,$$

where α is a costant.

A general dynamical system is given by $|x(t)\rangle = (x_1, x_2, ...x_n)^T$. Its evolution is given by

$$\frac{d}{dt}|x(t)\rangle = |f(|x(t)\rangle, t)\rangle$$

where **f** is a continuous and differentiable function. In terms of components the equations are

$$\begin{aligned}\dot{x_1} &= f_1(x_1, x_2, ..., x_n, t), \\ \dot{x_2} &= f_2(x_1, x_2, ..., x_n, t) \\ . &\quad . \\ \dot{x_n} &= f_n(x_1, x_2, ..., x_n, t),\end{aligned}$$

where f_i are continuous and differentiable functions. When the functions f_i are independent of time, the system is called *autonomous* system. However, when f_i are explicit function of time, the system is called *nonautonomous*. The three examples given above are autonomous sysetms.

A nonautonomous system can be converted to an autonomous oney by renaming $t = x_{n+1}$ and

$$\dot{x}_{n+1} = 1.$$

An example of nonautonomous system is

$$\begin{aligned}\dot{x} &= p \\ \dot{p} &= -x + F(t).\end{aligned}$$

The above system can be converted to an autonomous system using the following procedure.

$$\begin{aligned}\dot{x} &= p \\ \dot{p} &= -x + F(t) \\ \dot{t} &= 1.\end{aligned}$$

In the above examples, the system variables evolve with time, and the evolution is described using ordinary differential equation. There are however many situations when the system variables are fields in which case the evolution is described using partial differential equation. We illustrate these kinds of systems using examples.

1. **Diffusion Equation**
$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T.$$
Here the state variable is field $T(x)$. We can also describe $T(x)$ in Fourier space using Fourier coefficients. Since there are infinite number of Fourier modes, the above system is an infinite-dimensional. In many situations, finite number of modes are sufficient to describe the system, and we can apply the tools of nonlinear dynamics to such set of equations. Such systems are called low-dimensional models.

2. **Navier-Stokes Equation**
$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u}.$$
Here the state variables are $\mathbf{u}(\mathbf{x})$ and $p(\mathbf{x})$.

2.3 Continuous state variables and discrete time

Many systems are described by discrete time. For example, hourly flow Q_n through a pipe could be described by
$$Q_{n+1} = f(Q_n).$$
where f is a continuous and single-valued function, and n is the index for hour. Another example is evolution of the normalized population P_n is the population at nth year then
$$P_{n+1} = aP_n(1 - P_n).$$
Here the population is normalized with respect to the maximum population to make P_n as a continuous variable. Physically the first term represents growth, while the second term represents saturation.

The above equations are called **difference equations**.

Note that if the time gap between two observations become very small, then the description will be closer to continuous time case.

2.4 Discrete state variables and discrete time

For some ynamical systems the system variables are discrete, and they evolve in discrete time. A popular example is game of life where each site has a living cell or a dead cell. The cell at a given site can change from live to dead or vise versa depending on certain rules. For example, a dead cell becomes alive if number of live neighbours are between 3 to 5 (neither under-populated or over-populated). These class of dynamical systems show very rich patterns and behaviour. Unfortunately we will not cover them in this course.

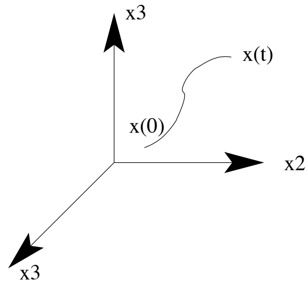
2.5 Discrete state variables and continuous time

The values of system variables of logic gates are discrete (0 or 1). However they depend on the external input that can exceed the threshold value in continuous time. Again, these class of systems are beyond the scope of the course.

In the present course we will focus on ordinary differential equations and difference equations that deal with continuous state variables but continuous and discrtet time respectively.

2.6 Nonlinear systems

2.7 State Space



A state space or phase space is an space whose axis are the state variables of a dynamical variables. The system's evolution can be uniquely determined from an initial condition in the state space.

Chapter 3

One Dimensional Systems

In this chapter we will consider autonomous systems with one variable. The evolution of this system will be described by a single first order differential equation (ODE). In this chapter we will study the dynamics of such systems.

3.1 Fixed points and local properties

The evolution equation of an autonomous one-dimensional dynamical system (DS) is given by

$$\dot{x} = f(x).$$

The points x^* at which $f(x^*) = 0$ are called "*fixed points*" (FP); at these points

$$\dot{x} = 0.$$

Hence, if at $t = 0$ the system at x^*, then the system will remain at x^* at all time in future. Noet that a system can have any number of fixed points (0, 1, 2, ...).

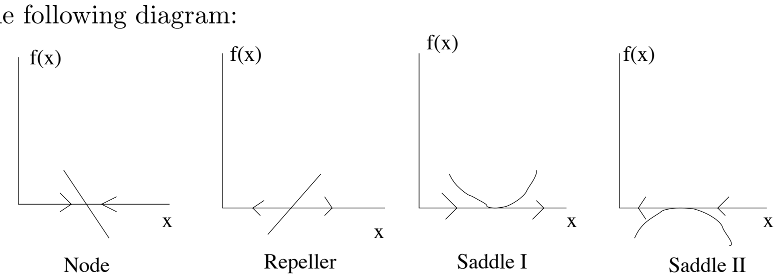
Now let us explore the behaviour of the DS near the fixed points:

$$\dot{x} \approx f'(x^*)(x - x^*),$$

whose solution is

$$x(t) - x^* = (x(0) - x^*) \exp\left(f'(x^*)t\right).$$

Clearly, if $f'(x^*) < 0$, the system will approach x^*. This kind of fixed point is called a *node*. If $f'(x^*) > 0$, the system will go away from x^*, and the fixed point is called a *repeller*. These two cases are shown in the first two figures of the following diagram:



Note that the motion is along the line (along x axis).

On rare ocassions $f(x^*) = f'(x^*) = 0$. In these cases, the evolution near the fixed point will be determined by the second derivative of f, i.e.,

$$\dot{x} \approx \frac{f''(x^*)}{2}(x - x^*)^2.$$

If $f''(x^*) > 0$ (third figure), then $\dot{x} > 0$ for both sides of x^*. If the system is to the left of x^*, it will tend towards x^*. On the other hand if the system is to the right of x^*, then it will go further away from x^*. This point is called *saddle point of Type I.* The reverse happens for $f''(x^*) < 0$, and the fixed point is called *saddle point of Type II.*

Examples

1. $\dot{x} = 2x$. The fixed point is $x^* = 0$. It is a repeller because $f'(0) = 2$ (positive).

2. $\dot{x} = -x+1$. The fixed point is $x^* = 1$. It is node because $f'(1) = -1$ (negative).

3. $\dot{x} = (x-2)^2$. The fixed point is $x^* = 2$. This point is a saddle point of Type I because $f''(2) = 2$ (positive).

The above analysis provides us information about the behaviour near the fixed points. So they are called *local behaviour of the system.*

3.2 Global properties

To understand the system completely, we need to understand the global dynamics as well. Fortunately, the global behaviour of 1D systems rather simple, and it can be easily deduced from the continuous and single valued nature of the function f. We illustrate the global behaviour using several examples.

Examples:

1. $\dot{x} = x(x-1)$. FPs are at $x = 0$ (node) & $x = 1$ (repeller). Using the single valued nature of $f(x)$ we can complete the state space diagram which is shown in Fig.

2. $\dot{x} = x(x-1)(x-2)$. The FPs are at $x = 0, 1, 2$. From the information on the slopes, we deduce that $x = 0$ and 2 are rellers and $x = 1$ is node. These information and continuity of $f(x)$ helps us complete the full state space picture., which is shown in Fig.

3. $\dot{x} = ax(1 - x/k)$ where a and k are positive constants. The fixed points are at $x = 0$ and k. Since $f'(0) = a > 0$, $x = 0$ is a repeller. For the other FP , $f'(k) = -a < 0$, so $x = k$ is a node. Using this information we sketch the state space, which is shown in Fig....

The above examples show how to make the state space plots for 1D systems.

Using the continuity and single-valued nature of the function $f(x)$ we can easily deduce the following global properties for a 1D dynamical system.

1. Two neighbouring FPs cannot be nodes or repeller.

2. Two repellers must have a node between them.

3. If the trajectories of a system are bounded, then the outermost FP along the x axis must be (i) node OR (ii) saddle I onthe left and saddle II on the right.

Note that the above properties are independent of the exact form of f. For example, systems $f(x) = x(x-1)$ and $f(x) = x(x-2)$ have similar behaviour even though the forms are different. These properties are called topological properties of the system.

Exercise

1. For the following systems, obtain the fixed points, and determine their stability. Sketch state-space trajectories and $x - t$ plot

   (a) $\dot{x} = ax$ for $a < 0$; $a > 0$.

   (b) $\dot{x} = 2x(1-x)$

   (c) $\dot{x} = x - x^3$

   (d) $\dot{x} = x^2$

   (e) $\dot{x} = \sin x$

   (f) $\dot{x} = x - \cos x$

Chapter 4

Two-dimensional Linear Systems

4.1 Fixed Points and Linear Analysis

A general two-dimensional autonomous dynamical system is given by

$$\begin{aligned}\dot{X}_1 &= f_1(X_1, X_2)\\ \dot{X}_2 &= f_2(X_1, X_2).\end{aligned}$$

The fixed points of the system are the ones where

$$\begin{aligned}f_1(X_1^*, X_2^*) &= 0\\ f_2(X_1^*, X_2^*) &= 0.\end{aligned}$$

The solution of the above equations yield fixed points which could be one or more.

Let us analyze the system's behaviour near the fixed point.We expand the functions $f_{1,2}$ near the fixed point (X_1^*, X_2^*). Using $X_1 - X_1^* = x_1$ and $X_2 - X_2^* = x_2$, the equation near the FP will be

$$\begin{aligned}\dot{x}_1 &= \frac{\partial f_1(X_1, X_2)}{\partial X_1}|_{(X_1^*, X_2^*)} x_1 + \frac{\partial f_1(X_1, X_2)}{\partial X_2}|_{(X_1,, X_2^*)} x_2\\ \dot{x}_2 &= \frac{\partial f_2(X_1, X_2)}{\partial X_1}|_{(X_1^*,, X_2^*)} x_1 + \frac{\partial f_2(X_1, X_2)}{\partial X_2}|_{(X_1^*, X_2^*)} x_2\end{aligned}$$

The four partial derivatives are denoted by a, b, c, d. The above equations can be written in terms of matrix:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{4.1}$$

Let us look at a trivial system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

whose solution is immediate:

$$\begin{aligned} x_1(t) &= x_1(0)\exp(at) \\ x_2(t) &= x_2(0)\exp(dt). \end{aligned}$$

When b and/or c nonzero, the equations get coupled. These equations however can be easily solved using matrix method described below.

4.2 Flows in the Linear Systems

In this section we will solve the equation (4.1) using similarity transformation. For the following discussion we will use "bra-ket" notation in which *bra* $\langle x|$ stands a row vector, while *ket* $|x\rangle$ stands for a column vector. In this notation, Eq. (4.1) is written as

$$|\dot{x}\rangle = A|x\rangle,$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the 2x2 matrix, and $|x\rangle = (x_1, x_2)^T$ is a column vector. The basic strategy to solve the above problem is to diagonalize the matrix A, solve the equation in the transformed basis, and then come back to the original basis.

As we will see below, the solution depends crucially on the eigenvalues λ_1, λ_2 of matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The eigenvalues of the matrix are

$$\lambda_{1,2} = \frac{1}{2}(Tr \pm \sqrt{Tr^2 - 4\Delta})$$

with $Tr = a + d$ is the trace, and $\Delta = ad - bc$ is the determinant of the matrix. These eigenvalues can be classified in four category

1. Real ones with $\lambda_1 \neq \lambda_2$ (when $Tr^2 > 4\Delta$).
2. Complex ones $\lambda_{\{1,2\}} = \alpha \pm \beta$ (when $Tr^2 < 4\Delta$).
3. $\lambda_1 = \lambda_2$ with $b = c = 0$ (here $Tr^2 = 4\Delta$).
4. $\lambda_1 = \lambda_2$ with $b \neq 0, c = 0$ (here $Tr^2 = 4\Delta$).

We will solve Eq. (4.1) for the four above cases separately.

4.2.1 Real Eigenvalues $\lambda_1 \neq \lambda_2$

The eigenvectors corresponding to the eigenvalues $\lambda_{1,2}$ are $|v_1\rangle = (1, -b/(a - \lambda_1)^T$ and $|v_2\rangle = (1, -b/(a - \lambda_2)^T$ respectively. Since $\lambda_1 \neq \lambda_1$, $|v_1\rangle$ and $|v_2\rangle$ are linearly independent. Using these eigenvectors we construct a nonsingular matrix S whose columns are $|v_1\rangle$ and $|v_2\rangle$. We denote the unit vectors by $|e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Clearly $S|e_1\rangle = |v_1\rangle$ and $S|e_2\rangle = |v_2\rangle$.

The matrix $S^{-1}AS$

$$S^{-1}AS|e_1\rangle = S^{-1}A|v_1\rangle = \lambda_1 S^{-1}|v_1\rangle = \lambda_1|e_1\rangle.$$

Similarly

$$S^{-1}AS|e_2\rangle = \lambda_2|e_2\rangle.$$

Therefore $S^{-1}AS$ is a diagonal matrix whose diagonal elements are the eigenvalues of A, i.e.,

$$S^{-1}AS = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

This procedure is called diagonalization of matrix. Note that the whole proof hinges on the existance of two linearly indendent eigenvectors.

In the following we will use the above theorem to solve the DE. Inversion of the above equation yields $A = SDS^{-1}$and

$$\dot{|x\rangle} = A|x\rangle = SDS^{-1}|x\rangle.$$

Using $S^{-1}|x\rangle = |y\rangle$, we obtain a much simpler looking equation

$$\dot{|y\rangle} = D|y\rangle,$$

whose solution is

$$|y(t)\rangle = y_1(0)\exp{(\lambda_1 t)}|e_1\rangle + y_2(0)\exp{(\lambda_2 t)}|e_2\rangle.$$

Using $|x\rangle = S|y\rangle$, we obtain

$$|x(t)\rangle = y_1(0)\exp{(\lambda_1 t)}|v_1\rangle + y_2(0)\exp{(\lambda_2 t)}|v_2\rangle.$$

We can derive the above solution in another way. The solution of Eq. (4.1) in the matrix form is

$$|x(t)\rangle = \exp{(At)}|x(0)\rangle.$$

Since

$$\exp{(At)} = S\exp{(Dt)}S^{-1},$$

we obtain

$$\begin{aligned}
\exp{(At)}|x(0)\rangle &= S\exp{(Dt)}|y(t)\rangle \\
&= S\exp{(Dt)}[y(0)|e_1\rangle + y_2(0)|e_1\rangle] \\
&= y_1(0)\exp{(\lambda_1 t)}|v_1\rangle + y_2(0)\exp{(\lambda_2 t)}|v_2\rangle,
\end{aligned}$$

which is same as the above result.

Graphically in y_1-y_2 coordinates the solution is

$$\begin{aligned}
y_1 &= y_1(0)\exp{(\lambda_1 t)} \\
y_2 &= y_2(0)\exp{(\lambda_2 t)}.
\end{aligned}$$

Elimination of t yields

$$y_2(t) = C[y_1]^{\lambda_2/\lambda_1},$$

where C is a constant. We can plot these the phase space trajectories in y_1-y_2 plane very easily. When $\lambda_{1,2}$ are of both positive (1,2 here), then we obtain curves of the type I (see Fig. 4.1). The fixed point is called a *repeller* and the phase space curves are $y_2 = Cy_1^2$. When the eigenvalues are both negative, the the fixed point is a *node*; the flow is shown in Fig. 4.2 (-1,-2 here) with phase

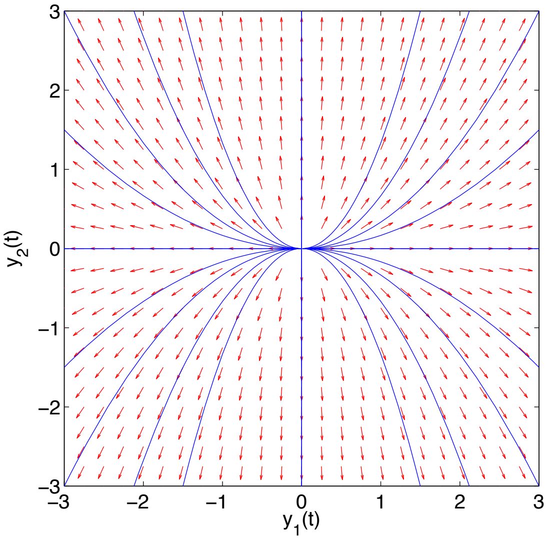


Figure 4.1: A phase space plot of a system $\dot{y_1} = y_1, \dot{y_2} = 2y_2$. The fixed point is a repeller.

space curves again as $y_2 = Cy_1^2$. In Fig. 3, λ_1 is positive, while $\lambda_2$is negative, with the fixed point termed as *saddle* (λ =1,-1 respectively). The flow diagram for a saddle is shown in Fig. 4.3 with the phase space curves as $y_2 y_1^2 = C$.

The $y_1$and $y_2$axes are the eigen directions in y_1-y_2 plane. If the initial condition lies on either of the axes, then the system will remain on that axis forever. Hence these are the invariant directions.

In x_1-x_2 plane the above mentioned state space plots are similar to those in y_1-y_2 plane except that the eigen directions are rotated. This is illustrated by the following example:

Example 1:

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$$

$Tr = -3$, $\Delta = 2$. The eigenvalues

$$\lambda_{1,2} = \frac{-3 \pm \sqrt{9-8}}{2} = -1, -2.$$

The eigenvectors corresponding to these values are

$$|v_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; |v_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

respectively.Hence

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

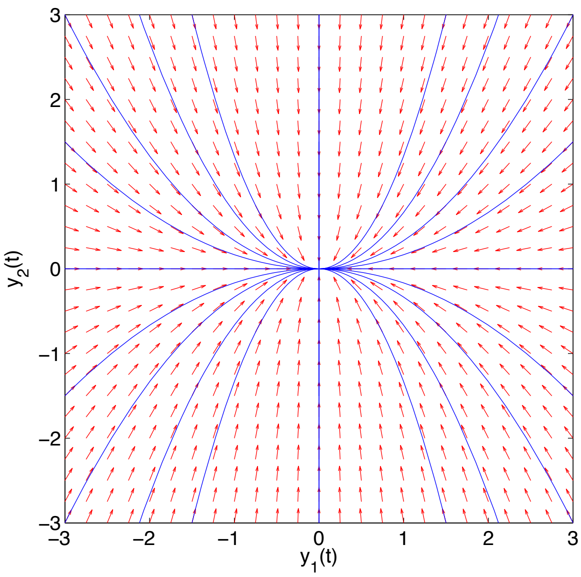


Figure 4.2: A phase space plot of a system $\dot{y_1} = -y_1, \dot{y_2} = -2y_2$. The fixed point is a node.

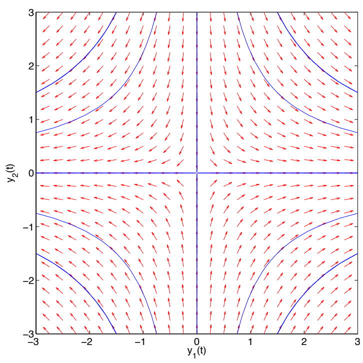


Figure 4.3: A phase space plot of a system $\dot{y_1} = y_1, \dot{y_2} = -2y_2$. The fixed point is a saddle.

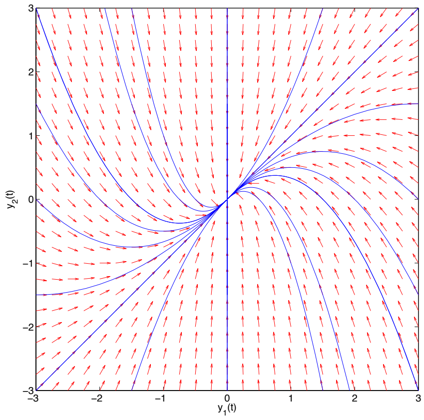


Figure 4.4: A phase space plot of a system $\dot{x_1} = -x_1, \dot{x_2} = x_1 - 2x_2$. The fixed point is a node.

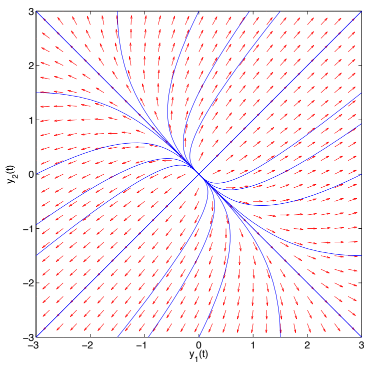


Figure 4.5: A phase space plot of a system $\dot{x_1} = 4x_1 + 2x_2, \dot{x_2} = 2x_1 + 4x_2$. The fixed point is a repeller.

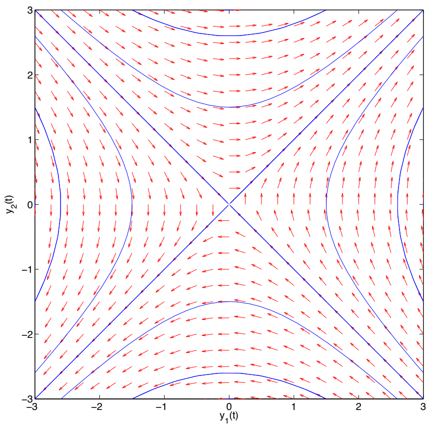


Figure 4.6: A phase space plot of a system $\dot{y_1} = y_2, \dot{y_2} = y_1$. The fixed point is a saddle.

It is easy to verify that

$$S^{-1}AS = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.$$

In the eigen basis the solution is

$$|y(t)\rangle = y_1(0)\exp(-t)|e_1\rangle + y_2(0)\exp(-2t)|e_2\rangle$$

which is shown in Fig. 4.2. The constants $y_{1,2}(0)$ can be obtained from the initial condition.

In the $x_1 - x_2$ basis the solution is

$$|x(t)\rangle = y_1(0)\exp(-t)|v_1\rangle + y_2(0)\exp(-2t)|v_2\rangle.$$

The flow in $x_1 - x_2$ is show in Fig. 4.4. If the initial condition lies on the eigen direction (either on $|v_1\rangle$ or $|v_2\rangle$), then the system will contiue to remain on the axis forever.

Example 2:

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

The eigenvalues of the matrix are 2 and 6, and the corresponding eigen vectors are $(-1,1)^T$ and $(1,1)^T$. The phase space picture in x_1-x_2 basis is shown in Fig. 4.5.

Example 3: Motion in a potential $U(x) = -x^2/2$

Equation of motion

$$\ddot{x} = -\frac{dU}{dx} = x.$$

Therefore,

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_1. \end{aligned}$$

Therefore the matrix A is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whose eigenvalues are 1 and -1. The corresponding eigenvectors are $|v_1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|v_2\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively. These are the diagonals as shown in Fig. 4.6. The eigenvalue is $+1$ long the eigen direction $|v_1\rangle$, so the system will move away from the origin as $\exp(t)$. However the system moves towards the the origin along $|v_2\rangle$ as $\exp(-t)$ since the eigenvalue is -1 along this direction.

In the $x_1 - x_2$basis the solution is

$$|x(t)\rangle = y_1(0)\exp(-t)|v_1\rangle + y_2(0)\exp(-t)|v_2\rangle.$$

After some manipulation we can show that

$$\begin{aligned} x_1 &= y_1 + y_2, \\ x_2 &= y_1 - y_2. \end{aligned}$$

One can easily show that the DEs are decoupled in $y_{1,2}$ variables.

We can eliminate t and find the equations of the curves

$$y_1 y_2 = C,$$

which are hyperbola. In terms of $x_1 - x_2$, the equations are

$$x_1^2 - x_2^2 = C'.$$

The flow are shown in Fig. 4.6.

The equation of the curves could also be obtained using

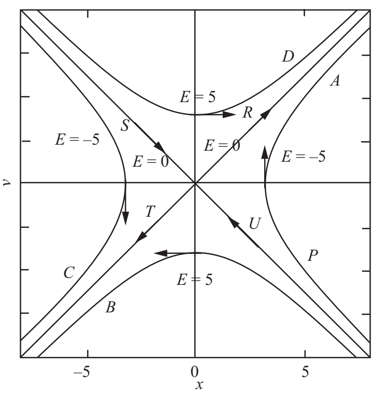
$$\frac{dp}{dx} = \frac{x}{p},$$

or

$$\frac{p^2}{2} - \frac{x^2}{2} = const = E.$$

The above equation could also be obtained from the conservation of energy.

The curves represent physical trajectories. See figure below. Note that $E < 0$ and $E > 0$ have very different bahviour. The curve $E = 0$ are the eigenvectors. Interpret physically. Show that when $E = 0$,the system take infinite time to reach to the top.



Our system is in region A at point P shown in the figure. See the directions of the arrow. The fixed point is $(0, 0)$. Technically this type of fixed point is called a **saddle**. All the unstable fixed points of mechnaical systems have this behaviour.

4.2.2 Complex Eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$

Let us solve the oscillator whose equation is

$$\ddot{x} = -x$$

or

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1. \end{aligned}$$

Clearly the matrix A is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

whose eigenvalues are $\pm i$. The corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ i \end{pmatrix}; \begin{pmatrix} 0 \\ -i \end{pmatrix}.$$

It is quite inconvenient to work with complex vectors. In the following discussion we will devise scheme to use real vectors to solve DEs whose eigenvalues are complex.

Along the first eigenvector the solution is

$$\begin{aligned} |x\rangle &= \exp(it) \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \\ &= |x_{re}\rangle + i|x_{im}\rangle \end{aligned}$$

The substitution of the above in $|\dot{x}\rangle = A|x\rangle$ easily yields

$$\begin{aligned} |\dot{x}_{re}\rangle &= A|x_{re}\rangle, \\ |\dot{x}_{im}\rangle &= A|x_{im}\rangle, \end{aligned}$$

Hence, $|x_{re}\rangle$ and $|x_{im}\rangle$ are both solution of the original DE. Since $|x_{re}\rangle$ and $|x_{im}\rangle$ are linearly independent solutions of the DE, we can write the general solution $\mathbf{x}(t)$ as a linear combination of $|x_{re}\rangle$ and $|x_{im}\rangle$:

$$\begin{aligned} |x\rangle &= c_1|x_{re}\rangle + c_2|x_{im}\rangle \\ &= x_1(0) \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + \dot{x}_1(0) \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} x_1(t) &= x_1(0)\cos t + x_2(0)\sin t \\ x_2(t) &= -x_1(0)\sin t + x_2(0)\cos t. \end{aligned}$$

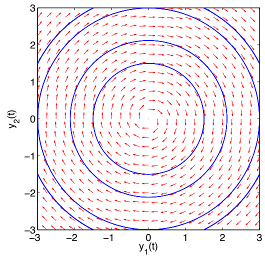
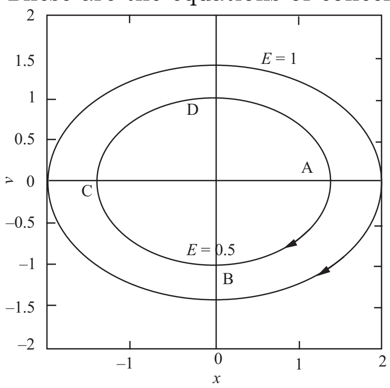


Figure 4.7: A phase space plot of a system $\dot{x_1} = x_2, \dot{x_2} = -x_1$. The fixed point is a node.

Note that $x_2(t) = \dot{x}_1$ is the velocity of particle. Both the position and velocity are periodic, as is expected for oscillatior. Clearly

$$x_1^2 + x_2^2 = c_1^2 + c_2^2 = C.$$

These are the equations of concentresic circles. See the following figure and



Note that we could have had derived the equations for the curve using the energy conservation.

Recall that the independent solutions for real eigenvalues were $\exp(\lambda_1 t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\exp(\lambda_2 t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, which corresponds to the motion along the eigen direction. For the above case with imaginary eigenvalues, the eigenvectors are not along a straight line in the real space. Here the system essentially moves as $\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ and $\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ which corresponds to the clockwise and anticlockwise circular motion respectively.

In the above discussion we have shown how to use real vectors to solve the DEs whose eigen values are pure imaginary. In the following we will extend the procedure to complex eigenvalues. Suppose the eigenvalues are $\alpha \pm i\beta$, and the eigenvector corresponding to $\alpha + i\beta$ is $|v\rangle = |v_1\rangle + i|v_2\rangle$ where $|v_1\rangle$ and $|v_2\rangle$ are real vectors. Using $A|v\rangle = (\alpha + i\beta)|v\rangle$ we obtain

$$\begin{aligned} A|v_1\rangle &= \alpha|v_1\rangle - \beta|v_2\rangle \\ A|v_2\rangle &= \beta|v_1\rangle + \alpha|v_2\rangle. \end{aligned}$$

Let

$$S = (|v_1\rangle \ \ |v_2\rangle),$$

then

$$\begin{aligned} S^{-1}AS|e_1\rangle &= \alpha|e_1\rangle - \beta|e_2\rangle \\ S^{-1}AS|e_2\rangle &= \beta|e_1\rangle + \alpha|e_2\rangle. \end{aligned}$$

Therefore

$$S^{-1}AS = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = B,$$

and the equations in terms of the transformed variables are

$$|\dot{y}\rangle = B|y\rangle. \tag{4.2}$$

Note that $|v^*\rangle$ is the other independent eigen vector. we can also diagonalize A by using $(\hat{\mathbf{v}}\ \hat{\mathbf{v}}^*)$. However we wish to avoid the complex vectors.

In the following discussion we will solve Eq. 4.2. In the new basis the eigenvector corresponding to $(\alpha + i\beta)$ is $\begin{pmatrix} 1 \\ i \end{pmatrix}$. Along this direction the evolution is

$$\begin{aligned} |y(t)\rangle &= \exp(\alpha t)\exp(i\beta t)\begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= \exp(\alpha t)\begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + i\exp(\alpha t)\begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix} \\ &= |y_{re}\rangle + i|y_{im}\rangle. \end{aligned}$$

We can immediately derive that

$$\begin{aligned} |\dot{y_{re}}\rangle &= B|y_{re}\rangle, \\ |\dot{y_{im}}\rangle &= B|y_{im}\rangle, \end{aligned}$$

Hence, $|y_{re}\rangle$ and $|y_{im}\rangle$ are both independent solution of Eq. 4.2, and the general solution $|y(t)\rangle$ as a linear combination of $|y_{re}\rangle$ and $|y_{im}\rangle$:

$$\begin{aligned} |y(t)\rangle &= c_1|y_{re}(t)\rangle + c_2|y_{im}(t)\rangle \\ &= x_1(0)\exp(\alpha t)\begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + x_2(0)\exp(\alpha t)\begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}. \end{aligned}$$

The equations of the trajectories are

$$\left[y_1^2 + y_2^2\right]\exp(-2\alpha t) = c_1^2 + c_2^2 = C,$$

which are the equations of spirals as shown in Fig. 4.8. When $\alpha < 0$, the system moves towards the origin (fixed point), and the fixed point is called **spiral node**. However when $\alpha > 0$, the system moves outward, and the fixed point is called a **spiral repeller**.

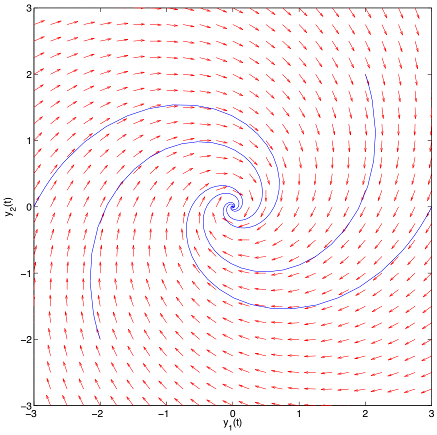


Figure 4.8: A phase space plot of a system $\dot{x_1} = -0.5x_1+x_2, \dot{x_2} = -x_1+0.5*x_2$. The fixed point is a spiral node.

4.2.3 Repeated eigenvalues

Consider matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The condition for the eigenvalues to be equal is $Tr^2 = 4\Delta$, which implies that

$$(a-d)^2 = -4bc. \tag{4.3}$$

If the eigenvector is $(u_1 u_2)^T$, then we obtain

$$\sqrt{-bc}u_1 + bu_2 = 0. \tag{4.4}$$

The above two conditions can be satisfied in the following cases:

1. $a = d$ and $b = c = 0$: Here any vector is an eigenvector and we can choose any two independent ones for writing the general solution of DE.
2. $a = d$, $c = 0$ but $b \neq 0$: Here only one eigenvector $(1, 0)^T$ exists.
3. $a = d$, $b = 0$ but $c \neq 0$: This is same as case 2.
4. $a \neq d$: Here both b and c are nonzero, hence there will only be one eigenvector.

The above four cases essentially fall into two cases: (a) having two indendent eigenvectors; (b) haveing only one eigenvector.In the following discussions we will study the solution of $|\dot{x}\rangle = A|x\rangle$ under the above two cases.

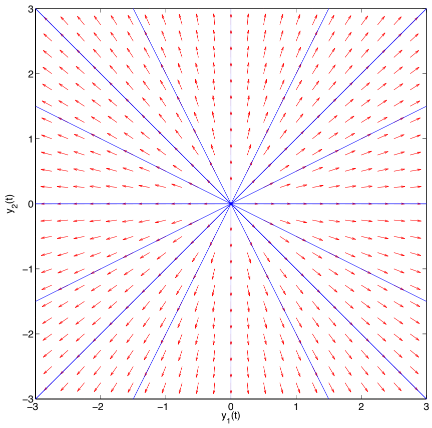


Figure 4.9: A phase space plot of a system $\dot{x_1} = 2x_1, \dot{x_2} = 2x_2$. The fixed point is a repeller.

With two independent eigenvectors: $\lambda_1 = \lambda_2, b = c = 0$

According to the above discussion, the matrix A will be of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI.$$

The solution is trivial.

$$\mathbf{x}(t) = x_1(0)\exp{(at)}\mathbf{E}_1 + x_2(0)\exp{(at)}\mathbf{E}_2.$$

The equation of the curves are

$$x_2(t) = Cx_1(t)$$

as shown in Fig. 4.9. Note that anytwo linearly independent eigenvectors are eigenvectos of the matrix.

With only eigenvector: $\lambda_1 = \lambda_2, b \neq 0, c = 0$

Using Cayley–Hamilton theorem theorem we have

$$(A - \lambda I)^2 |w\rangle = 0 \tag{4.5}$$

for all $|w\rangle$. Suppose the one eigenvector is $|v\rangle$ then

$$(A - \lambda I)|v\rangle = 0.$$

We can expand any vector in a plane using $|v\rangle$ and another linearly independent vector, say $|e\rangle$. Hence

$$|w\rangle = \alpha|v\rangle + \beta|e\rangle.$$

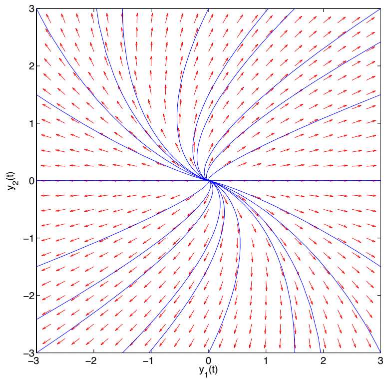


Figure 4.10: A phase space plot of a system $\dot{y_1} = 2y_1 + y_2, \dot{y_2} = 2y_2$. The fixed point is a repeller.

Substitution of the above form of $|w\rangle$ in Eq. (4.5) yields

$$(A - \lambda I)|e\rangle = \mu|v\rangle.$$

If $\mu = 0$, $|e\rangle$ will also be a eigenvector, which is contrary to our assumption. Hence $\mu \neq 0$. Using $|e\rangle/\mu = |u\rangle$, we obtain

$$A|u\rangle = |v\rangle + \lambda|u\rangle.$$

Using

$$S = [|v\rangle \; |u\rangle]$$

we obtain

$$S^{-1}AS = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = B,$$

which is the Jordon-Cannonical form for the 2x2 matrix with repeated eigenvalues. In the new basis

$$\begin{aligned} \dot{y}_1 &= \lambda y_1 + y_2 \\ \dot{y}_2 &= \lambda y_2 \end{aligned}$$

whose solution is

$$|x(t)\rangle = \begin{pmatrix} \exp(\lambda t) & t\exp(\lambda t) \\ 0 & \exp(\lambda t) \end{pmatrix}.$$

We summarize various fixed points using Tr-Δ plots.

4.3 Damped linear oscillator

The equation of a damped linear oscillator is

$$\ddot{x} + 2\gamma\dot{x} + x = 0$$

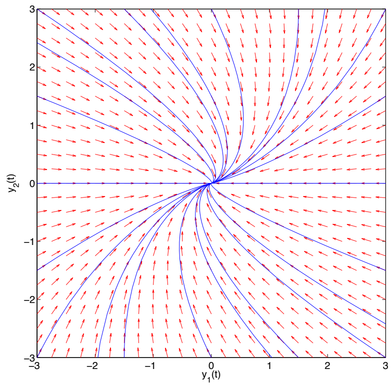


Figure 4.11: A phase space plot of a system $\dot{y_1} = -2y_1 + y_2, \dot{y_2} = -2y_2$. The fixed point is a node.

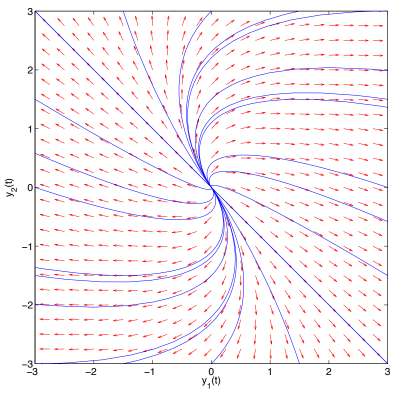


Figure 4.12: A phase space plot of a system $\dot{x_1} = 3x_1 + x_2, \dot{x_2} = -x_1 + x_2$. The fixed point is a repeller.

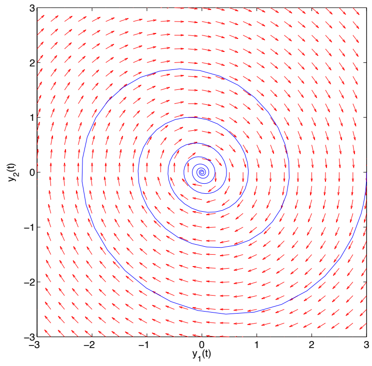


Figure 4.13: A phase space plot of a system $\dot{x_1} = x_2, \dot{x_2} = -x_1 + 2\gamma x_2$ with $\gamma = 0.1$. The fixed point is a spiral node.

which reduced to

$$\begin{aligned}\dot{x_1} &= x_2 \\ \dot{x_2} &= -x_1 - 2\gamma x_2.\end{aligned}$$

The eigenvalues for the system are

$$\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - 1}.$$

Clearly

- For $\gamma < 1$, the eigenvalues are complex.
- For $\gamma > 1$, the eigenvalues are real and negative
- For $\gamma = 1$, the eigenvalues are real and repeated.

The state space plots for these three cases are shown in the following three figures. Find out the eigenvectors for these cases.

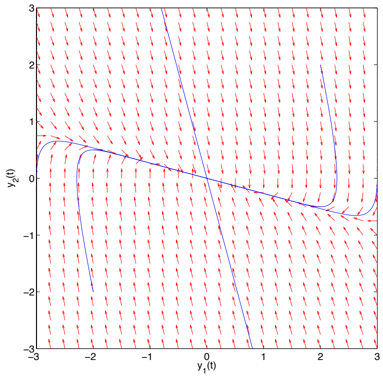


Figure 4.14: A phase space plot of a system $\dot{x_1} = x_2, \dot{x_2} = -x_1 + 2\gamma x_2$ with $\gamma = 2$. The fixed point is a node.

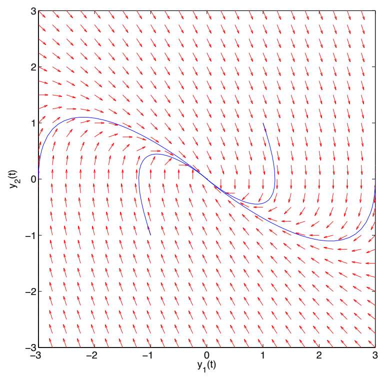


Figure 4.15: A phase space plot of a system $\dot{x_1} = x_2, \dot{x_2} = -x_1 + 2\gamma x_2$ with $\gamma = 1$. The fixed point is a node.

Chapter 5

Conjugacy of the Dynamical Systems

Suppose the two linear systems $\dot{\mathbf{x}} = A\mathbf{x}$ and $\dot{\mathbf{y}} = B\mathbf{y}$ have flows $\phi^A$and ϕ^B respectively. These two systems are (topologically) conjugate if there exists a homeomorphism $h : R^2 \to R^2$that satisfies

$$\phi^B(t, h(\mathbf{X}_0)) = h(\phi^A(t, \mathbf{X}_0)).$$

The homeomorphism h is called a conjugacy. Thus a conjugacy takes the solution curves of $\dot{\mathbf{x}} = A\mathbf{x}$ to $\dot{\mathbf{y}} = B\mathbf{y}$.

Example:

Two systems $\dot{x} = \lambda_1 x$ and $\dot{y} = \lambda_2 y$.

We have flows

$$\phi^j(t, x_0) = x_0 \exp(\lambda_j t)$$

for $j = 1, 2$. If $\lambda_{1,2}$are nonzero and have the same sign, then

$$h(x) = \begin{cases} x^{\lambda_2/\lambda_1} \text{for } x \leq 0 \\ -|x|^{\lambda_2/\lambda_1} \text{ for x<0} \end{cases}$$

Hence the two systems are conjugate to each other. This works only when $\lambda_{1,2}$ have the same sign.

5.1 Linear systems

Def: A matrix A is hyperbolic if none of its eigenvalues has real part 0. We also say that the system $\dot{\mathbf{x}} = A\mathbf{x}$ is hyperbolic.

Thm: Suppose that the 2×2 matrices $A_1$and $A_2$are hyperbolic. Then the linear systems $\dot{\mathbf{x}} = A\mathbf{x}$ are conjugate if and only if each matrix has the same number of eigenvalues with negative real parts.

Without proof.

Chapter 6

2D Systems: Nonlinear Analysis

The nonlinear system is given by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{f}(\mathbf{x})$ is a nonlinear function. In 2D we write

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

with f and g as nonlinear functions of x and y. Let us denote the domain by D. $D \in R^2$.

6.1 Global Picture

6.1.1 Example 1: Pendulum

The nondimensionalized equation of a pendulum is

$$\ddot{\theta} = -\sin\theta,$$

where θ is the angle from the stable equilibrium position in the anticlockwise direction. We can rewrite the above equation as

$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -\sin\theta. \end{aligned}$$

The DS has two fixed points: $(0, 0)$ and $(\pi, 0)$ (because the system is periodic in θ).

Linearization near $(0, 0)$ yields

$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -\theta. \end{aligned}$$

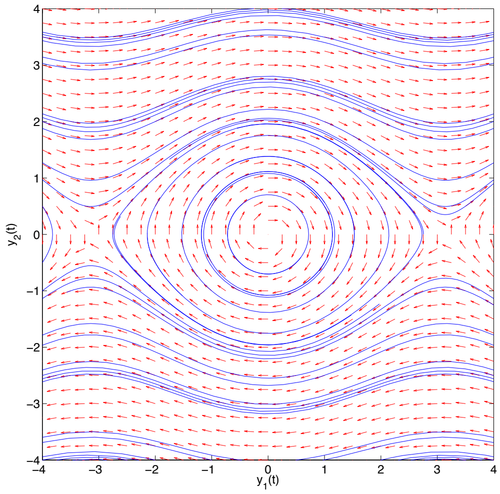


Figure 6.1: A phase space plot of a system $\dot{y_1} = y_1, \dot{y_2} = 2y_2$. The fixed point is a repeller.

Clearly the fixed point is a center. It is consistent with the fact that the $(0,0)$ is a stable equilibrium point. Linearization near $(0,\pi)$yields

$$\begin{aligned}\dot{\phi} &= v \\ \dot{v} &= \phi,\end{aligned}$$

which is a saddle. Consistent because $(\pi, 0)$ is an unstable equilibrium point. The eigen vectors at $(\pi, 0)$ are (1,1) and $(1, -1)$ (evs. 1, -1). Sketch the linear profile.

Now let us try to get the global picture. The conservation of energy yields

$$\frac{v^2}{2} + (1 - \cos\theta) = E$$

or

$$v = \pm\sqrt{2(E - 1 + \cos\theta}.$$

The above function can be plotted for various values of E. For $E = 0$ we get a point $(0,0)$. For $E = 1$ we get the curves $v = \pm\sqrt{2\cos\theta}$ which passes through the saddle point. These curves are called **separatrix** for the reasons given below. For $E < 1$, the curves are closed and lie between the two separatrix. Above $E = 1$, the curves are open as shown in the figure. The separatrix separate the two set of curves with differen qualitative behaviour.

Q: How long does it take for the pendulum to reach to the top when $E = 1$.

Note that the above curves nicely join the lines obtained using linear analysis. Something similar happens for the following system as well.

6.1.2 Example 2

The equation of motion is

$$\ddot{x} = -x + x^3.$$

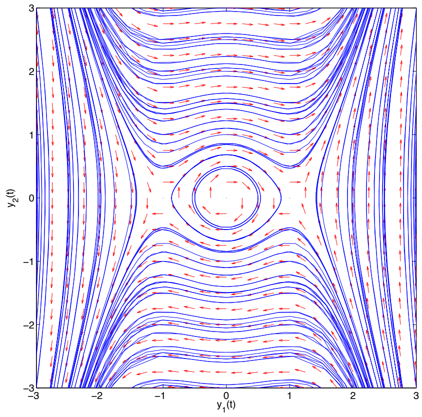
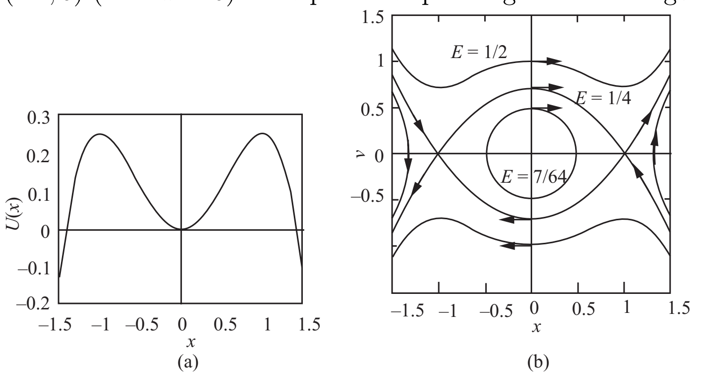


Figure 6.2: A phase space plot of a system $\dot{y_1} = y_1, \dot{y_2} = 2y_2$. The fixed point is a repeller.

The force ($\dot{p}$) is zero at $x = 0, \pm 1$. Therefore, the fixed point are $(0,0)$ and $(\pm 1, 0)$ (note $\dot{x} = 0$). The potential plot is given in the figure:



Clearly $(0,0)$ is a stable FP, and $(\pm 1, 0)$ are unstable FP. Near $(0,0)$, the equation are

$$
\begin{aligned}
\dot{x} &= p, \\
\dot{p} &= -x,
\end{aligned}
$$

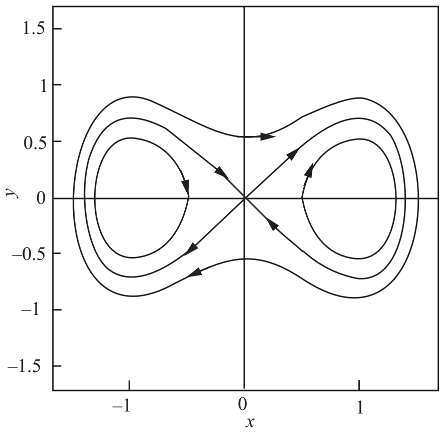
which is the equation of the oscillator. Hence, near $(0,0)$ the behaviour will be the same as that near the oscillator. Now near $(1,0)$, change the variable to $x' = x - 1$. In terms of (x', p), the equations are

$$
\begin{aligned}
\dot{x} &= p, \\
\dot{p} &= 2x',
\end{aligned}
$$

which is the equation for the unstable hill discussed in Problem 3.5.5. Hence, the phases space around $(1,0)$ should look like of Problem 3.5.5. The same thing for $(-1,0)$ also. You can see from the potential plot that $(\pm 1, 0)$ are unstable

points. Therefore, the phase space plot will look like as shown in the figure. What are the other phase trajectories doing?
(b) By similar analysis we draw the phase space trajectory for a DS whose equation is

$$\ddot{x} = x - x^3.$$



6.2 Invariant Manifolds

A Manifold is a subspace of the state space that satisfies continuity and differentiability property. For example, fixed points, x-axis, etc. are manifolds. Among these, **invariant manifolds** are special. If the initial condition of a DS starts from an point on a manifold and stays within it for all time, then the manifold is invariant. For linear systems, the fixed points and eigen vectors are invariant manifolds. The system tends to move away from the fixed points on the eigenvectors corresponding to the positive eigenvalues, hence these eigenvectors are **unstabl**e **manifold**. The eigenvectors with negative eigenvectors are **stable manifold** since the system tends to move toward the fixed points if it starts on these curves.

For nonlinear systems, the fixed points are naturally invariant manifolds. However the stable and unstable manifolds are not the eigenvectors of the linearized matrix of the fixed points. Yet we can find the stable and unstable manifolds for the nonlinear systems using the following definitions.

Def: The stable manifold is the set of initial conditions $|x_0\rangle$ such that $|x(t)\rangle \to |x^*\rangle$ as $t \to \infty$ where $|x^*\rangle$ is the fixed point. Similarly unstable manifold is the set of initial conditions $|x_0\rangle$ such that $|x(t)\rangle \to |x^*\rangle$ as $t \to -\infty$ (going backward in time).

For the pendulum and $x^2 - x^4$ potential (Figs), the fixed points and the constant energy periodic curves are invariant manifolds. In addition stable and unstable manifolds for the saddle are also visible. Incidently, the unstable manifold of one saddle merges with the stable manifold of the other saddle. Trajectories of these kind that join two different saddles are called **heteroclinic trajectories** or **saddle connection**. For $-x^2+x^4$ potential (Fig), the unstable manifold of the saddle merges with its own stable manifold; such trajectories are called **homoclinic trajectories (orbits)**.

6.3 Stability

A FP is **stable equilibrium** if for every neighborhood O of $x*$ in R^nthere is a neighbourhood O_1 of x^* such that every solution $x(t)$ with $\mathbf{x}(0) = \mathbf{x}_0$in $O_1$is defined and remains in O for all $t > 0$. This condition is also called **Liapunov stability.**

In addition, if $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*$, then the FP is called **asymptotically stable** (or **attracting**).

The FPs which are not stable are called unstable FP.

Examples and some important points to remember

- Center- stable but not asymptotically stable
- Node- asymptotically stable
- Saddle, repellers- unstable
- For nonlinear systems, the stability is difficult to ascertain.
- Liapunov stability does not imply asymptotic stability (e.g., center). When a system is Liapunov stable but not asymptotic stable, it is called neutrally stable.
- Asymptotic stability does not imply Liapunov stability. Consider a DS $\dot{\theta} = 1 - \cos\theta$. The asymptotic fixed point is $\theta = 0$, but $f'(\theta = 0) = 0$ and $f''(\theta = 0) > 0$. So the systems is approaches $\theta = 0$ from left,. However for any initial condition $\theta > 0$, the system increases to $\theta = 2\pi$ and reaches the fixed point due to its periodic nature.

6.4 No Intersection Theorem and Invariant Sets

Two distinct state space trajectories cannot intersect in a finite time. Also, a single trajectory cannot cross itself.

Proof: Given an initial condition, the future of the system is unique. Given this we can prove the above theorem by a proof of contradiction. It the state space trajectories interesect in a finite time, then we can take the point of intersection to be the initial point for the future evolution. Clearly, at the point of intersection there will be two directions of evolution, which is a contradiction. Hence No Intersection Theorem.

Note that the trajectories can intersect at infinite time. This point is a saddle.

6.5 Linear vs. Nonlinear

Hartman-Grobman Theorem: Suppose the $n-$dimensional system $\dot{\mathbf{x}} = F(\mathbf{x})$ has an equilibrium point at $\mathbf{x}_0$ that is hyperbolic. Then the nonlinear flow is conjugate to the flow of the linearized system in a neighbourhood of $\mathbf{x}_0$.

In addition, there exist local stable and unstable manifolds $W^s_{loc}(\mathbf{x}_0)$ and $W^u_{loc}(\mathbf{x}_0)$ of the same dimension n_s and n_u as those of the eigenspaces E^s and E^u of the linearized equations, and the manifolds are tangent to E^u and E^s.

6.5.1 Examples

(1) DS

$$\begin{aligned}\dot{x} &= x + y^2\\ \dot{y} &= -y\end{aligned}$$

Locally: saddle
General solution

$$\begin{aligned}x(t) &= (x_0 + \frac{1}{3}y_0^2)\exp t - \frac{1}{3}y_0^2\exp(-2t)\\ y(t) &= y_0\exp -t\end{aligned}$$

Sketch the plot. The stable manifold is the same as E^s. However the unstable manifold is

$$x + y^2/3 = 0.$$

Locally the same behaviour as the linearized system.
(2) DS

$$\begin{aligned}\dot{x} &= -y + x(\mu - r^2)\\ \dot{y} &= x + y(\mu - r^2)\end{aligned}$$

Linear spiral, nonlinear-

$$\begin{aligned}\dot{r} &= r(\mu - r^2)\\ \dot{\theta} &= 1.\end{aligned}$$

Same behaviour near the fixed point.
(3) DS

$$\begin{aligned}\dot{x} &= -y + \epsilon x r^2\\ \dot{y} &= x + \epsilon y r^2.\end{aligned}$$

The linear solution- center.
Nonlinear eqns

$$\begin{aligned}\dot{r} &= \epsilon r^3\\ \dot{\theta} &= - 1.\end{aligned}$$

which is a spiral. So near the FP the linear behaviour si very different then the nonlinear behaviour.
(4) DS

$$\begin{aligned}\dot{x} &= x^2\\ \dot{y} &= -y\end{aligned}$$

The nonlinear soln

$$\begin{aligned}x(t) &= \frac{x_0}{1 - x_0 t}\\ y(t) &= y_0\exp(-t)\end{aligned}$$

The $y$axis is the stable manifold.
The linear behaviour is $x(t) = const$. The nonlinear and linear bahaviour very different.
The above examples illustrate Hartman-Grobman theorem.

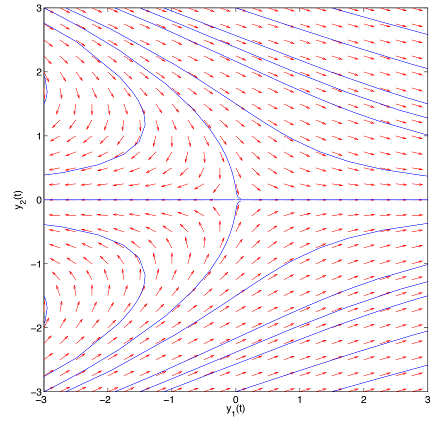


Figure 6.3: A phase space plot of Example 1

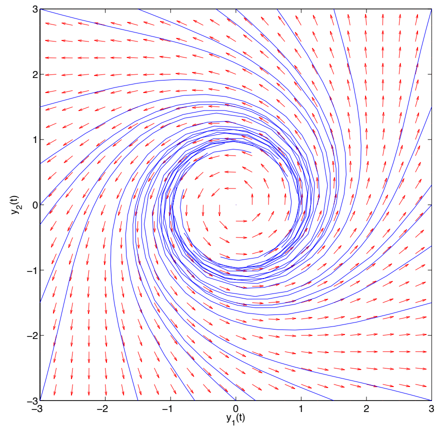


Figure 6.4: A phase space plot of Example 3; outward spiral

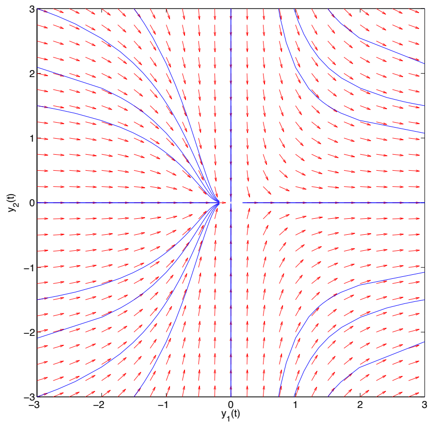


Figure 6.5: A phase space plot of Example 4

6.6 Dissipation and The Divergence Theorem

Let us look at the time evolution of an area. Consider an are given in the figure.

It's area is

$$A = dxdy = (x_B - x_A)(y_D - y_C).$$

By differentiating the above, we obtain

$$\begin{aligned}\frac{dA}{dt} &= (x_B - x_A)[g(x_D, y_D) - g(x_C, y_C)] + [f(x_B, y_B) - f(x_A, y_A)](y_B - y_A) \\ &= dxdy\frac{\partial g}{\partial y} + dxdy\frac{\partial f}{\partial x}.\end{aligned}$$

Hence,

$$\frac{1}{A}\frac{dA}{dt} = div(f, g).$$

The above theorem can be easily generalized to any dimenison as

$$\frac{1}{V}\frac{dV}{dt} = \nabla \cdot \mathbf{f}.$$

If the $div < 0$, the the volume will shrink, and if the $div > 0$, then the volume will grow. The flows with $div = 0$ are area preserving. The systems with div<0 are called dissipative systems, and ones which are area preserving are called Hamiltonian systems.

Examples:

1. Show that all the mechanical systems which can be described by position-dependent potentials are area-preserving. Demonstrate using SHM as an example.

2. Show that state-space of frictional oscillator (positive friction) is dissipative.

3. Consider

$$\begin{aligned}\dot{x} &= \sin x(-0.1\cos x - \cos y)\\ \dot{y} &= \sin y(\cos x - 0.1\cos y).\end{aligned}$$

Describe the motion.

Note that the div is trace of matrix A for the linear system or linearized system near the fixed point. Since trace is invariant under similarity transformation:

$$div(f,g) = Tr(A) = \lambda_1 + \lambda_2.$$

6.7 Poincare-Bendixon's Theorem

6.7.1 Bendixon's Criterion

Suppose that the domain $D \in R^2$ is simply connected (no 'holes' or 'separate parts' in the domain) and f and g continuously differentiable in D . The system can only have periodic solutions if $\nabla \cdot (f,g) = 0$ or if it changes sign. If the div is not identically zero or it does not changes sign in D, then the system has no closed solution lying entirely in D.

Proof: If we have a closed orbit C in D. The interior of D is G. Then Gauss law yields

$$\int_G \nabla \cdot (f,g) d\sigma = \int_C (f dy - g dx) = \int_C (f\frac{dy}{dt} - g\frac{dx}{dt})dt = 0,$$

which is possible only if $\nabla \cdot (f,g) = 0$ everywhere or if it changes sign. If $\nabla \cdot (f,g)$ has one sign throughout R, then the above condition will not be satisfied, hence no closed orbit will be allowed. Note that this is a necessary condition, but not a sufficient condition.

Examples

1. Van der Pol oscillator

   $$\ddot{x} + x - \mu(1-x^2)\dot{x}$$

   with μ as a constant. The DS can be rewritten as

   $$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x + \mu(1-x^2)y.\end{aligned}$$

   Clearly the divergence is $\mu(1-x^2)$. For $|x| < 1$, the div is nonzero and monotonic, hence this region cannot have a periodic solution in the domain |x|<1 .

2. A DS

   $$\begin{aligned}\dot{x} &= -x + y^2\\ \dot{y} &= -y^3 + x^2.\end{aligned}$$

   $div = -1 - 3y^2$, which is clearly negative. Hence DS does not have a periodic solution.

3. SHM: div=0. Periodic solution may exist. In fact, the system has infinite periodic orbits.

4. For Hamiltonian systems

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\frac{dU}{dx}\end{aligned}$$

where U is the potential. Clearly $\nabla\cdot(f,g)=0$. But Hamiltonian system need not have a closed orbit. For example, $U=-x^2$. This example illustrates that Bendixon's criterion is a necessary condition for the existance of closed orbit, but not a sufficient condition.

6.7.2 Poincare-Bendixon's Theorem

Consider a two-dimensonal system $|\dot{x}\rangle = |f(|x\rangle)\rangle$ in a region R with $|f(|x\rangle)\rangle = (f,g)^T$ being a continuous and differentiable vector field. If R is a **closed** and **bounded** subset of the plane, and if a trajectory C is confined in R, then either

- C is a closed orbit;
- C approached a closed orbit asymptotically as $t\to\infty$;
- C approaches a fixed point asymptotically as $t\to\infty$.

If the DS does not have a fixed point then the system is guaranteed to have an isolated closed orbit called **limit cycle**.

Example:

$$\begin{aligned}\dot{x} &= -y + x(\mu - r^2)\\ \dot{y} &= x + y(\mu - r^2)\end{aligned}$$

with $\mu>0$. Bounded system. 0 is a repeller spiral. Since there is no other fixed point, the system must have a LC.

Example A DS is given by

$$\begin{aligned}\dot{x} &= x(x^2+y^2-2x-3)-y\\ \dot{y} &= y(x^2+y^2-2x-3)+x.\end{aligned}$$

Hence the div is

$$\begin{aligned}div &= 3(x^2+y^2)+y^2+x^2-6x-3-3\\ &= 4\left[(x-\frac{3}{4})^2+y^2-\frac{33}{16}\right].\end{aligned}$$

The bracketed term is an equation of a circle with center at $(3/4,0)$ and radius $\sqrt{33}/4$. The sign of div is negative for all the points within the circle. So, using Bendixon's criterion, there cannot be a periodic orbit within the circle.

In radial polar coordinate

$$\begin{aligned}\dot{r} &= r(r^2-2r\cos\theta-3)\\ \dot{\theta} &= 1.\end{aligned}$$

The term in the bracket is positive for $r>3$, mixed for $1<r<3$, and negative for $r<1$. Then the annulus $1<r<3$ should have a limit cycle. It will be an unstable LC.

6.8 No chaos in 2D systems

Chaos- divergence of trajectories in bounded systems.

No Intersection thm. prohibits the diverging trajectories to come back. Hence no chaos in 2D. The diverging trajectories keeps diverging and moving away.

6.9 Ruling out closed orbits

Strogatz Sec. 7.2 p199.

- Gradient system
- Liapunov function
- Energy function