I Semester M.Sc. Examination, May 2015
MATHEMATICS
Algebra

Time : 3 Hours Max. Marks : 80

Note : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Let H and K be finite subgroups of a group G. Then prove that $|HK| = \frac{0(H)0(K)}{0(H \cap K)}$. 5

 b) State and prove Lagrange theorem for finite groups. 5

 c) Define a commutator subgroup. If G' is a commutator subgroup of a group G then prove that G' is normal in G and G/G' is abelian. 6

2. a) Let H and N be subgroups of a group G and let N be normal in G. Then prove the following : 8

 i) HN is a subgroup of G

 ii) N is normal in HN

 iii) $H \cap N$ is normal in H

 iv) $H/_{H \cap N} \cong HN/_N$

 b) State and prove Cayley's theorem. 8

3. a) Derive the class equation. 6

 b) If G is a finite group and p^r divides 0(G), where p is a prime, then prove that G has a subgroup of order p^r. 6

 c) Show that any group of order $5^2.7^2$ is abelian. 4

4. a) if $f : R \to S$ is an onto ring homomorphism with ker f = I. Then prove that $R/_I \cong S$. 7

 b) Define a prime ideal. Prove that P is a prime ideal of $\mathbb{Z}$ if and only if P = (0) or P = (p) for some prime p. 5

 c) Let a and b be non-zero elements of a ring R. Then prove that a and b are associates if and only if a = ub for some unit U in R. 4

P.T.O.

5. a) Prove that a nonzero element p of a ring R is irreducible if and only if (p) is maximal among all proper principal ideals of R. **4**
   b) Prove that in a unique factorization domain, an element is prime if and only if it is irreducible. **8**
   c) If F is a field then prove that F[X] is an euclidean domain. **4**

6. a) State and prove Eisensten's criterion. **8**
   b) If F, K and L are fields such that K is algebraic over F and L is algebraic over K then prove that L is algebraic over F. **8**

7. a) Let $f(X) \in F[X]$ be of degree n. Then prove that there is an extension K of F which is a splitting field of f(X) and the degree of K over F is atmost n !. **8**
   b) Find the splitting field of $X^3 - 2$ over Q. **4**
   c) Let $f(X) \in F[X]$ be an irreducible polynomial. Then prove that f(X) has a multiple root if and only if Df(X) is a zero polynomial. **4**

8. a) Prove that the multiplicative group of non-zero elements of a finite field is cyclic. **6**
   b) State and prove the primitive element theorem. **10**

MATH 1.2

I Semester M.Sc. Examination, May 2015
MATHEMATICS
Real Analysis – I

Time : 3 Hours Max. Marks : 80

Note : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Define an ordered field. If x, y $\in$ IR such that $0 < x < y$ then prove that $0 < \frac{1}{y} < \frac{1}{x}$. 3

   b) State and prove the Archimedean property. 5

   c) Prove that for every real x > 0 and every integer n > 0, there is one and only one real y such that $y^n = x$. 8

2. a) Define a countable set. Prove that every subset of a countable set is countable. 8

   b) Prove that the set of all sequences whose elements are the digits 0 and 1 is uncountable. 8

3. a) Let X be a metric space and $E \subset X$. If p is a limit point of E then prove that every neighbourhood of p contains infinitely many points of E. 8

   b) Prove that the interval [a, b] is compact in IR'. 8

4. a) Prove that a sequence $\{x_n\}$ converges to x if and only if every subsequence of $\{x_n\}$ converges to x. 8

   b) If $\{x_n\}$ and $\{y_n\}$ are sequences converging to x and y, respectively, then prove that $\lim_{n\to\infty} (x_n + y_n) = x + y$. 4

   c) Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. 4

5. a) Prove that a monotonically increasing sequence is convergent if and only if it is bounded. 7

   b) Show that '$\mathcal{e}$' is irrational. 4

   c) Prove that every bounded sequence of real numbers contains a convergent subsequence. 5

P.T.O.

6. a) Let $a_1 \geq a_2 \geq \ldots \geq 0$. Then prove that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots$ converges. **8**

b) Discuss the convergence of the following series :

i) $\sum_{n=0}^{\infty} \frac{1}{n!}$ ii) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ iii) $\sum_{n=0}^{\infty} \frac{n^3+1}{2^n+1}$ **(2+3+3)**

7. a) State and prove Kummer's test and hence prove Raabe's test. **8**

b) Let $\sum a_n$ be a series of positive terms. Suppose $\lim_{n\to\infty} \left(n \log \frac{a_n}{a_{n+1}}\right) = \alpha$. Then prove that

i) $\sum a_n$ converges if $\alpha > 1$ and

ii) $\sum a_n$ diverges if $\alpha < 1$. **8**

8. a) State and prove Abel's test. **8**

b) State and prove Merten's theorem. **8**

MATH 1.3

I Semester M.Sc. Examination, May 2015
MATHEMATICS
Complex Analysis – I

Time : 3 Hours Max. Marks : 80

Instructions : *1) Answer **any five** questions.*
*2) **Each** question carries **equal** marks.*

1. a) State and prove Lagrange's identity in the complex form.

   b) Prove that $\left|\dfrac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| = 1$ if either $|z_1| = 1$ or $|z_2| = 1$. What exception must be made if $|z_1| = |z_2| = 1$? **(10+6)**

2. a) Prove that the points z_1, z_2, will be inverse points with respect to the circle $z_1 \bar{z}_2 + \bar{\alpha} z_1 + \alpha \bar{z}_2 + r = 0$.

   b) Show that the triangle whose vertices are the points represented by the complex numbers z_1, z_2, z_3 on the argand diagram is equilateral if and only if $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ **(6+10)**

3. a) State and prove necessary and sufficient conditions for a function to be analytic.

   b) Show that an analytic function with constant modulus is constant. **(12+4)**

4. a) Show that the sum function f(z) of the power series $\sum_{n=0}^{\infty} a_n Z^n$ represents an analytic function at every point inside the circle of convergence of the power series. Further more, every power series possesses derivatives which are obtained through term by term differentiation of the series.

   b) Find the analytic function f(z) = u + iv given that $u - v = e^x (\cos y - \sin y)$. **(10+6)**

5. a) Show that any bilinear transformation can be expressed as a product of transformation, rotation, magnification or contraction and inversion.

   b) Find the bilinear transformation which maps the points $z_1 = 0$, $z_2 = -i$ and $z_3 = 1$ into $w_1 = i$, $w_2 = 1$ and $w_3 = 0$ respectively.

   c) If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining points (1, 1) and (2, 3). Find the values of $\int_C (12z^2 - 4iz)dz$. **(6+4+6)**

P.T.O.

6. a) If z_1 and z_2 are any two points in R (a simply connected region), then show that $\int_{z_1}^{z_2} f(z)\,dz$ is independent of the path in R joining z_1 and z_2.

b) Let f(z) be analytic in the region bounded by two simple closed curves c and c_1 (where c_1 lies inside c). Then show that $\int_{c} f(z)dz = \int_{c_1} f(z)dz$, where c_1 and c are in counter clockwise direction.

c) Evaluate $\int_c \frac{dz}{(z-i)(z+i)}$, where c is the circle

i) $c : |z + i| = 1$ ii) $c : |z - i| = 1$ **(6+6+4)**

7. a) If f(z) is analytic in the entire complex plane and bounded, then show that f(z) is a constant function.

b) State and prove fundamental theorem of algebra.

c) Evaluate $\int_c \frac{z-3}{z^2+2z+5}dz$, where c is the circle $|z + 1 - i| = 2$. **(6+6+4)**

8. a) Suppose that a function f is analytic in an open disk Δ and that γ is a closed piecewise smooth path in Δ. Then prove that $\eta(\gamma, z)\, f(z) = \int_\gamma \frac{f(\xi)d\xi}{\xi - z}$ for every z in Δ/γ where $\eta(\gamma, z)$ is the winding number of γ about z.

b) Find the residue of

i) $\frac{1}{(z^2+1)^3}$ at $z = i$ and ii) $\frac{z^2}{z^2+a^2}$ at $z = ia$. **(10+6)**

I Semester M.Sc. in Mathematics Examination, May 2015
DISCRETE MATHEMATICS

Time : 3 Hours Max. Marks : 80

Note : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) What are the conditional statements ? Explain with an example. 4

b) Show that :

i) $(p \to q) \vee (\sim p \to q)$ is a tautology

ii) $\sim (p \wedge q) \wedge (\sim p \vee \sim q)$ is a contingency

iii) $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction

iv) $(p \leftrightarrow q) \vee (\sim p \to q)$ is a contingency 12

2. a) Define disjunctive normal form, conjunctive normal form and principle disjunctive normal form. Obtain the principle disjunctive normal form of $(\sim p \vee \sim q) \to (\sim p \wedge q)$ 6

b) i) Prove the following statement by an indirect proof : if n^2 is an even integer, then n is an even. 5

ii) Symbolize and negate the following statement if some straight lines intersect, then all straight lines are not parallel. 5

3. a) Establish the following result by using Mathematical induction. 6

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

b) State well ordering, pigeon-hole and extended pigeon-hole principles. 5

c) Show that if seven numbers from 1 to 12 choosen, then two of them will add upto 13. 5

4. a) A total amount of 1500 is to be distributed among three students A, B and C. In how many ways distribution can be made in multiples of 100 if (i) Everybody must get atleast 300 (ii) A must get atleast 500 and B, C should not get atleast 400. 6

b) Find the generating function for the number of partitions of a +ve integer n of the form $n = x_1 + 2x_2 + 5x_3$, given $x_1, x_2, x_3 \geq 1$. Hence find the number of partition for 12. **8**

c) Define catalan numbers and find the co-efficient of x° in $\left(3x^2 - \frac{2}{x}\right)^{15}$. **2**

5. a) Explain the method of solving the recurrence relation by using generating function. **10**

b) Solve the recurrence relation $a_{n+1} - a_n = n^2$ for $n \geq 0$, and $a_0 = 1$. **6**

6. a) Solve the recurrence relation using difference method.

i) $u_{x+2} - 10u_{x+1} + 20u_x = 2^x + 10$. **5**

ii) $u_{n+2} + u_{n+1} + u_n = n^2 + n + 1$. **6**

b) Solve $F_{n+2} = F_{n-1} + F_n$ for $n \geq 0$, given $f_0 = 0$, $f_1 = 1$. **5**

7. a) Prove that every equivalence relation defined on a set give rise to a partition of the set into equivalence class, conversely any partition a set determines an equivalence relation R such that the numbers of the partition are precisely the equivalence class defined by R. **10**

b) Prove that, Let R be a relation on a set A, then R^{∞} is the transitive closure of R. **6**

8. a) Define poset with an example. **8**

Let A = {1, 2, 3, 4, 5, 6, 7 8 } be the poset, then write Hasse diagram and discuss its l.u.b. and g.l.b.

b) Define Lattice with an example. Let $(L, \leq)$ be a Lattice, then show that **8**

$$a \leq b \Rightarrow \begin{cases} a \vee c \leq b \vee c \\ a \wedge c \leq b \wedge c \end{cases} \quad \forall a, b, c \in L$$

First Semester M.Sc. (Mathematics) Examination, May 2015
DIFFERENTIAL EQUATIONS

Time : 3 Hours Max. Marks : 80

*Note : 1) Answer **any five** questions.*
*2) **No** additional sheets will be provided for answering.*
*3) Use of **scientific calculator** is **permitted**.*

1. a) State the Lipschitz condition for a function f defined on closed domain D of the xy-plane. Give an example of a function (i) which satisfy Lipschitz condition and (ii) which do not satisfy Lipschitz condition. Justify your answer.

   b) Define fundamental set. Prove that a set of n solutions $\{\phi_j(x) : j = 1, 2, \ldots n\}$ of linear differential equation $L_n y = 0$ on some interval I forms a fundamental set of solutions if and only if Wronskian $W\{(\phi_j(x) : j = 1, 2, \ldots n\} \neq 0$ for all $x \in I$. **(8+8)**

2. a) Define Wronskian of real valued functions $f_1(x), f_2(x), \ldots f_n(x)$ defined on an interval I. If $\{\phi_j(x) : j = 1, 2, \ldots n\}$ are n solutions of differential equation $a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n y = 0$ on some internal I, then for any x, $x_0 \in I$ prove that, $W\{\phi_j(x) : j = 1, 2, \ldots n\} = W\{\phi_j(x_0) : j = 1, 2, \ldots n\}$
$$\exp\left\{-\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)}\,dt\right\}.$$

   b) State and prove Sturm's comparison theorem. **(6+10)**

3. a) Describe the method of variation of parameter to solve n^{th} order linear differential equation $a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n y = b(x)$. Using this method find the solution of $x^2y'' - xy' - 3y = x^3$.

   b) Find eigen values and eigen functions of Sturm Loivelli problem $y'' + \lambda y = 0$, $y(0) = y(\pi) = 0$. **(10+6)**

4. a) Write the general form of Laguerre differential equation. Discuss its series solution and define its polynomials.

   b) Prove that Laguerre polynomials are orthogonal with respect to weight function e^{-x} on $(0, \infty)$. (8+8)

5. a) Find the general solution of following homogeneous linear equation.

$$\frac{dX}{dt} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 4 & 1 & -4 \end{pmatrix} X$$

   b) Define an autonomous system of differential equation. Define critical point, trajectories, of a system of differential equations. Given examples. (10+6)

6. a) Describe the method to find the nature of critical points, stability and paths of the non-linear system of equations $\frac{dx}{dt} = P(x, y), \frac{dy}{dt} = Q(x, y)$, where P(x, y) and Q(x, y) have continuous partial derivatives.

   b) Solve the following Cauchy problem $u_x + u_y + u - 1 = 0$ with the initial data $u = \sin x$ on $y = x + x^2$. (8+8)

7. a) Define complete integral of a non-linear first order partial differential equation. Find the complete integral of the equation $pq - p - q = 0$.

   b) Explain Jacobi's method to finding complete integral of partial differential equation. Using the method find the complete integral of $u_x x^2 - u_y^2 - a u_z^2 = 0$. (8+8)

8. a) Explain the canonical transformation of solving second order partial differential equation.

   b) Classify the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 u}{\partial y \partial x} = 0$ and reduce it to canonical form. (8+8)